# Mathematical Quantum Mechanics

TMP Programme, Munich – Winter Term 2010/2011

**EXERCISE SHEET 8**, issued on Tuesday 14 December 2010
**Due:** Tuesday 11 January 2011 by 8,15 a.m. in the designated "MQM" box on the 1st floor
**Info:** `www.math.lmu.de/~michel/WS10_MQM.html`

*Each exercise is worth a full mark as specified below. Correct answers without proofs are not accepted. Each step should be justified. You can hand in the solutions either in German or in English. The exercise marked with ★ is for extra credit.*

**Exercise 29 [13 points].** (*Unpolarised light*) In the lecture the photon considered was in a specific polarisation state. Normally, light is emitted by light sources as a mixture of photons in different polarisation states. How do we describe a single photon in such a beam of light quantum-mechanically? For simplicity, the light beam is assumed to be mixed from photons of two sources, where each source emits photons in some specific polarisation state $|\psi_1\rangle$ or $|\psi_2\rangle$ respectively. A photon is supposed to come with probability $p_1$ from source 1, with probability $p_2$ from source 2. $p_1 + p_2 = 1$.

(i) Show that the expectation value for the spin of a photon from this beam (the measurement result averaged over many photons) is given by

$$p_1\langle\psi_1|\hbar S|\psi_1\rangle + p_2\langle\psi_2|\hbar S|\psi_2\rangle$$

(*Hint:* With which probability is an individual photon left- or right-circular polarised?)

(ii) What is the expectation value for the spin of a photon in state

$$|\psi\rangle = \alpha|\psi_1\rangle + \beta|\psi_2\rangle?$$

Use this to show that one cannot describe a mixture of different polarisations by photons that are all in this state. How would one have to "relax" the form of the above state (i.e. vary from photon to photon), such that the result of (i) is achieved after averaging over many photons? Give a physical interpretation.

(iii) Unpolarised light is light that is found in any arbitrary polarisation state with equal probability (remark: there is also partially polarised light, which is neither in a specific polarisation state nor equally distributed over all of them). Show that light is unpolarised if light is for one (not: any, that would be the definition!) orthonormal basis $\{|\psi_1\rangle, |\psi_2\rangle\}$ with equal probability in state $|\psi_1\rangle$ or $|\psi_2\rangle$.

**Exercise 30 [12 points].** (*Density matrix: foundations*)

(i) Assume a photon in a (normalised) state $|\psi\rangle$. We define

$$P_\psi := |\psi\rangle\langle\psi|.$$

Show that the expectation value of a physical quantity given by operator $Q$ reads

$$\mathrm{Tr}\ \mathrm{P}_\psi \mathrm{Q}.$$

The matrix $P_\psi$ is called the *density matrix* of the *pure state* $|\psi\rangle$.

(ii) Assume the photon to be in a "state" that consists of a mixture of a (normalised) state $|\psi_1\rangle$ with probability $p_1$, a (normalised) state $|\psi_2\rangle$ with probability $p_2$, and so on, with $\sum_i p_i = 1$. Define

$$\rho := \sum_i p_i |\psi_i\rangle\langle\psi_i|.$$

Show that for this *mixed state* the expectation value of a physical quantity $Q$ is given by

$$\langle Q\rangle = \mathrm{Tr}\ \rho \mathrm{Q}.$$

The matrix $\rho$ is called the *density matrix* of a *mixed state.*

(iii) Show: (a) $\rho^\dagger = \rho$; (b) the eigenvalues of a density matrix are real and non-negative; (c) the sum of the eigenvalues of a density matrix is 1; (d) $\rho^2 = \rho \Leftrightarrow$ the state is pure.

★ **Exercise 31 [15 points].** (*Density matrix – Light*) In this exercise you may use the results of Exercises 29 and 30 even if you could not derive them.

(i) Write down the density matrix for unpolarised light (*Hint:* the definition in Exercise 29 (iii).) How could one determine from polarisation measurements whether a ray is unpolarised? Is there a minimum number of necessary measurements?

(ii) Write down the density matrix for a mixed state that consists 50 percent of linearly polarised light in $x$-direction and 50 percent of right-circular polarised light. Determine two orthogonal states that yield the same density matrix.

**Exercise 32 [15 points].** (*Spin in a Rotating Magnetic Field*) A particle with spin-$\frac{1}{2}$ and gyromagnetic ratio $\gamma$ is exposed to a time-dependent magnetic field

$$\underline{B}(t) = B_o\underline{e}_z + B_1(\cos(\omega t)\underline{e}_x + \sin(\omega t)\underline{e}_y).$$

The dynamics of the spin is determined by the Hamilton operator

$$H(t) = -\frac{\hbar\gamma}{2}\underline{\sigma}\cdot\underline{B}(t).$$

$|\uparrow\rangle$ and $|\downarrow\rangle$ are the eigenstates of $\sigma_z$ with eigenvalues $+1$ and $-1$.

(i) Solve the time-dependent Schrödinger equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle$$

with initial condition $|\psi(0)\rangle = |\uparrow\rangle$. Use the ansatz

$$|\psi(t)\rangle = U(t)|\tilde{\psi}(t)\rangle \quad \text{with} \quad U(t) = \exp\left(-\frac{i}{2}\omega t\sigma_z\right).$$

Give an interpretation of the ansatz.

(ii) What is the probability $P_\downarrow(t)$ to have the system in state $|\downarrow\rangle$ at time $t > 0$, if it was in state $|\uparrow\rangle$ at time $t = 0$? Discuss the dependency of the probability $P_\downarrow(t)$ on $\omega$.