EXERCISE SHEET 7, issued on Tuesday 7 December 2010
Due: Tuesday 14 December 2010 by 8,15 a.m. in the designated "MQM" box on the 1st floor
Info: `www.math.lmu.de/~michel/WS10_MQM.html`

Each exercise is worth a full mark as specified below. Correct answers without proofs are not accepted. Each step should be justified. You can hand in the solutions either in German or in English. The exercise marked with ★ is for extra credit.

Exercise 25 [13 points]. Decide which of the following sequences in $L^1_{\rm loc}(\mathbb{R})$ converge in $\mathcal{D}'(\mathbb{R})$ and compute the limit when it exists.

(i) $\{f_n\}_{n=1}^\infty$ with $f_n(x) := \dfrac{n}{\pi(1+n^2x^2)}$

(ii) $\{g_n\}_{n=1}^\infty$ with $g_n(x) := \dfrac{\sin nx}{\pi x}$

(iii) $\{h_n\}_{n=1}^\infty$ with $h_n(x) := n^2 x\cos(nx)$

(iv) $\{k_n\}_{n=1}^\infty$ with $k_n(x) := \Big(\dfrac{n^2}{1+n^2(nx-1)^2}\Big)^2$.

★ Exercise 26 [15 points]. Consider the Hamiltonian of the harmonic oscillator in one dimension, i.e., $H = -\dfrac{d^2}{dx^2} + x^2$. Consider the O.D.E. associated to the eigenvalue problem $H\psi = E\psi$, i.e.,

$$-\psi''(x) + x^2\psi(x) \;=\; E\psi(x) \qquad\qquad (\forall x\in\mathbb{R}) \qquad\qquad (\bullet)$$

with $E\in\mathbb{R}$ and $\psi\in C^2(\mathbb{R})$ to be determined.

(i) Consider the function $u\in C^2(\mathbb{R})$ defined by $\psi(x) = e^{-x^2/2}u(x)$ where $\psi\in C^2(\mathbb{R})$ is a solution to $(\bullet)$ for some $E\in\mathbb{R}$. Show that $(\bullet)$ is equivalent to the following O.D.E. for u:

$$u''(x) - 2xu'(x) + (E-1)u(x) \;=\; 0 \qquad\qquad (\forall x\in\mathbb{R})\,. \qquad\qquad (\bullet\bullet)$$

(ii) Determine the general solution $u\in C^2(\mathbb{R})$ to $(\bullet\bullet)$. (*Hint:* u of the general form $u(x) = \sum_{j=0}^\infty a_j x^j$, $a_j\in\mathbb{R}$.)

(iii) Deduce from (i) and (ii) that for $\psi\in L^2(\mathbb{R})\cap C^2(\mathbb{R})$ to solve $(\bullet)$, necessarily $E = E_n := 2n+1$ for some $n = 0,1,2,\dots$

(iv) For any non-negative integer n define

$$H_n(x) \ := \ (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \qquad x \in \mathbb{R}\,.$$

Prove the recursive relations

$$H_n' = 2xH_n - H_{n+1} \quad (n = 0, 1, 2, \dots) \qquad \text{and} \qquad H_n' = 2nH_{n-1} \quad (n = 1, 2, \dots).$$

(*Hint:* for the second one, you may consider the function $G(x,t) := e^{-t^2+2tx}$ and its Taylor expansion $G(x,t) = \sum_{n=0}^{\infty} a_n(x) t^n$, and use the property $\partial_x G = 2tG$.)

(v) Deduce from (i), (iii), and (iv) that for any non-negative integer n the O.D.E.

$$-\psi''(x) + x^2\psi(x) \ = \ (2n+1)\psi(x) \qquad\qquad (x \in \mathbb{R})$$

has a unique (normalised) solution in $L^2(\mathbb{R}) \cap C^2(\mathbb{R})$, given by $\psi = \psi_n$ where

$$\psi_n(x) \ := \ \frac{1}{\pi^{1/4}\sqrt{2^n\, n!}}\, e^{-\frac{1}{2}x^2} H_n(x)\,.$$

(vi) Show that $\{\psi_n\}_{n=0}^{\infty}$ defined in (v) is an orthonormal basis of $L^2(\mathbb{R})$.

Exercise 27 [12 points]. Consider the energy $\mathcal{E}[\psi]$ of the harmonic oscillator in the state ψ and the ground state energy E_0, i.e.,

$$\mathcal{E}[\psi] \ := \ \int_{-\infty}^{\infty} \Big(|\psi'(x)|^2 + x^2|\psi(x)|^2 \Big)\, \mathrm{d}x \qquad \text{and} \qquad E_0 := \inf_{\substack{\psi \in \mathcal{S}(\mathbb{R}) \\ \|\psi\|_2 = 1}} \mathcal{E}[\psi]\,.$$

Let ψ_{GS} be a (normalised) ground state wave-function of the one-dimensional harmonic oscillator, i.e., $\mathcal{E}[\psi_{\mathrm{GS}}] = E_0$, $\|\psi_{\mathrm{GS}}\| = 1$. Prove that $E_0 = 1$ and that ψ_{GS} is unique and is given by $\psi_{\mathrm{GS}}(x) = \pi^{-1/4} e^{-x^2/2}$. Give two alternative proofs.

(i) Use the orthonormal basis $\{\psi_n\}_{n=0}^{\infty}$ of $L^2(\mathbb{R})$ (see Exercise 26 (vi)) and the expansion $\psi_{\mathrm{GS}} = \sum_{n=0}^{\infty} a_n \psi_n$.

(ii) Prove directly that $\mathcal{E}[\psi] \geqslant \|\psi\|_2^2 \ \forall \psi \in \mathcal{S}(\mathbb{R})$. (*Hint:* use that $\int |\psi' + x\psi|^2 \geqslant 0$.)

Exercise 28 [15 points]. Recall the definition of the Sobolev space $H^\alpha(\mathbb{R}^d)$ (where d is a positive integer and $\alpha > 0$):

$$H^\alpha(\mathbb{R}^d) \ := \ \Big\{ f \in L^2(\mathbb{R}^d) \mid \|f\|_{H^\alpha}^2 := \int_{\mathbb{R}^d} \big(1 + (2\pi|k|)^{2\alpha}\big) |\widehat{f}(k)|^2 \,\mathrm{d}k \ < \ \infty \Big\}\,.$$

Assume that $\alpha > \frac{d}{2} + \ell$ for some non-negative integer ℓ. Prove that $H^\alpha(\mathbb{R}^d) \subset C^\ell(\mathbb{R}^d)$.