Mathematical Quantum Mechanics

TMP Programme, Munich – Winter Term 2010/2011

EXERCISE SHEET 4, issued on Tuesday 16 November 2010
Due: Tuesday 23 November 2010 by 8,15 a.m. in the designated "MQM" box on the 1st floor
Info: `www.math.lmu.de/~michel/WS10_MQM.html`

> *Each exercise is worth a full mark as specified below. Correct answers without proofs are not accepted. Each step should be justified. You can hand in the solutions either in German or in English.*

Exercise 13. Consider the hydrogenic Hamiltonian $H^{(Z)} = -\Delta - \dfrac{Z}{|x|}$ $(Z > 0)$ in three dimensions. Recall that its ground state energy is

$$\mathcal{E}_{\mathrm{GS}}(Z) := \inf_{\substack{\psi\in\mathcal{M}\\ \|\psi\|_2=1}} \langle\psi, H^{(Z)}\psi\rangle = -\frac{Z^2}{4}$$

where $\mathcal{M} = \{\psi \,|\, \psi, \nabla\psi, |\cdot|^{-1/2}\psi \in L^2(\mathbb{R}^3)\}$. Consider the coherent states of the form $\psi_{q,p,\theta}$ where (see Exercise 4)

$$\psi_{q,p,\theta}(x) := \frac{1}{(\theta\sqrt{\pi})^{3/2}}\, e^{ipx}\, e^{-\frac{|x-q|^2}{2\theta^2}}\,, \qquad x \in \mathbb{R}^3$$

for some q and p in $\mathbb{R}^3$ and $\theta > 0$. Estimate $\mathcal{E}_{\mathrm{GS}}(Z)$ from above by minimising the energy on the coherent states only and prove that

$$\inf_{q,p\in\mathbb{R}^3,\theta>0} \langle\psi_{q,p,\theta}, H^{(Z)}\psi_{q,p,\theta}\rangle = -\frac{2Z^2}{3\pi}\,.$$

Exercise 14.

(i) Prove that

$$\int_{\mathbb{R}^3} \frac{|\psi(x)|^2}{|x|^2}\,\mathrm{d}x \leqslant 4\,\|\nabla\psi\|_2^2 \qquad \forall\psi \in C_0^\infty(\mathbb{R}^3)\,.$$

(*Hint:* you may prove it by analogy with the proof given in class of the inequality)

$$\int_{\mathbb{R}^3} \frac{|\psi(x)|^2}{|x|}\,\mathrm{d}x \leqslant \|\nabla\psi\|_2\|\psi\|_2 \qquad \forall\psi \in H^1(\mathbb{R}^3) \qquad (\clubsuit)$$

i.e., computing commutators.)

(ii) Prove by a standard density argument that

$$\int_{\mathbb{R}^3} \frac{|\psi(x)|^2}{|x|^2}\,\mathrm{d}x \;\leqslant\; 4\,\|\nabla\psi\|_2^2 \qquad\qquad \forall\psi\in H^1(\mathbb{R}^3)\,. \qquad (\spadesuit)$$

(iii) Deduce from ($\spadesuit$) the following inequality

$$\int_{\mathbb{R}^3} \frac{|\psi(x)|^\alpha}{|x|}\,\mathrm{d}x \;\leqslant\; 2^\alpha \|\nabla\psi\|_2^\alpha\,\|\psi\|_2^{2-\alpha} \qquad\qquad \forall\psi\in H^1(\mathbb{R}^3)$$

for any $\alpha\in[0,2]$. (Note that for $\alpha=1$ one obtains ($\clubsuit$) apart from the right constant.)

Exercise 15. Denote by $\mathbb{1}_{\{|x|\leqslant 1\}}$ the characteristic function of the ball of unit radius in $\mathbb{R}^3$. Denote by $\chi_<$ the cut-off function $\chi_<(x) := \xi(|x|)\ \forall x\in\mathbb{R}^3$ where $\xi:\mathbb{R}^+\to[0,1]$ is smooth and such that $\xi(r)=1\ \forall r\in[0,\frac{1}{2}]$ and $\xi(r)=0\ \forall r\geqslant 1$. Use the definition of the Sobolev space H^1 (formula (11.28) in the handout "*Crash course in Analysis*") to prove that

(i) the function $\dfrac{\chi_<}{|\cdot|^\alpha}$ belongs to $H^1(\mathbb{R}^3)$ if and only if $\alpha<\frac{1}{2}$

(ii) the function $\mathbb{1}_{\{|x|\leqslant 1\}}$ does not belong to $H^1(\mathbb{R}^3)$.

Exercise 16. Same assumption as in Exercise 15. Use the Fourier characterisation of the Sobolev space H^1 (Theorem 11.3 in the handout "*Crash course in Analysis*") to prove that

(i) the function $\dfrac{\chi_<}{|\cdot|^\alpha}$ belongs to $H^1(\mathbb{R}^3)$ if and only if $\alpha<\frac{1}{2}$

(ii) the function $\mathbb{1}_{\{|x|\leqslant 1\}}$ does not belong to $H^1(\mathbb{R}^3)$.