**EXERCISE SHEET 3**, issued on Tuesday 9 November 2010
**Due:** Tuesday 16 November 2010 by 8,15 a.m. in the designated "MQM" box on the 1st floor
**Info:** `www.math.lmu.de/~michel/WS10_MQM.html`

*The full mark in each exercise is 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in the solutions either in German or in English.*

**Exercise 9.** Let $\{\chi_j\}_{j=1}^m$ be a family of bounded functions in $C^\infty(\mathbb{R}^3)$ such that $\sum_{j=1}^m \chi^2(x) = 1$ $\forall x \in \mathbb{R}^3$. Prove the following identity of operators on $\mathcal{S}(\mathbb{R}^d)$:

$$-\Delta \;=\; \sum_{j=1}^{m} \big(\, \chi_j(-\Delta)\chi_j - |\nabla \chi_j|^2 \,\big)\,.$$

**Exercise 10.** Consider the two Hamiltonians

$$H \;=\; -\Delta - \frac{1}{|x|}\,, \qquad \text{and} \qquad H^{(x_0)} = -\Delta - \frac{1}{|x|} - \frac{1}{|x-x_0|}$$

acting on wave-functions of variable $x \in \mathbb{R}^3$, for some $x_0 \in \mathbb{R}^3$, $x_0 \neq 0$. Define their ground state energy as

$$\mathcal{E}_{\mathrm{GS}} \;:=\; \inf_{\substack{\psi\in\mathcal{M}\\ \|\psi\|_2=1}} \langle \psi, H\psi\rangle \qquad \text{and} \qquad \mathcal{E}_{\mathrm{GS}}^{(x_0)} \;:=\; \inf_{\substack{\psi\in\mathcal{M}\\ \|\psi\|_2=1}} \langle \psi, H^{(x_0)}\psi\rangle$$

respectively, where $\mathcal{M} = \{\psi \,|\, \psi, \nabla\psi, |\cdot|^{-1/2}\psi \in L^2(\mathbb{R}^3)\}$. (Recall that $\mathcal{E}_{\mathrm{GS}} = -\frac{1}{4}$.)

(i) Prove that

$$\mathcal{E}_{\mathrm{GS}}^{(x_0)} \;\leqslant\; \mathcal{E}_{\mathrm{GS}} - \frac{1}{2}\, e^{-|x_0|} \qquad\qquad (\forall x_0)\,.$$

(*Hint:* ground state wave-function of the Hydrogen atom as a trial function.)

(ii) Prove that there exist constants $c, R > 0$ such that

$$\mathcal{E}_{\mathrm{GS}}^{(x_0)} \;\geqslant\; \mathcal{E}_{\mathrm{GS}} - \frac{c}{|x_0|} \qquad\qquad \forall |x_0| \geqslant R\,.$$

(*Hint:* formula from Exercise 9.)

**Exercise 11.**

(i) Show that the large time asymptotics of the free evolution of a wave-function $\psi \in L^2(\mathbb{R}^d)$ is given by

$$\Big\| e^{it\Delta}\psi - \frac{e^{i\frac{(\cdot)^2}{4t}}}{(4\pi i t)^{d/2}}\,\widehat{\psi}\Big(\frac{\cdot}{4\pi t}\Big)\Big\|_2 \xrightarrow{|t|\to\infty} 0\,.$$

(ii) Assume in addition that $|\cdot|^2\psi \in L^2(\mathbb{R}^d)$. Show that the above asymptotics takes the following quantitative form (for $|t|$ sufficiently large):

$$\Big\| e^{it\Delta}\psi - \frac{e^{i\frac{(\cdot)^2}{4t}}}{(4\pi i t)^{d/2}}\,\widehat{\psi}\Big(\frac{\cdot}{4\pi t}\Big)\Big\|_2 \leqslant \frac{\|\,|\cdot|^2\psi\|_2}{|t|}\,.$$

(iii) Assume now that $|\cdot|^r(D^s\psi) \in L^2(\mathbb{R}^d)$ for $r,s = 0,1,\dots,k$, $r+s\leqslant k$, where $k$ is a given positive integer (for instance, this is the case when $\psi \in \mathcal{S}(\mathbb{R}^d)$). Show that asymptotically as $|t|\to\infty$ the free evolution $e^{it\Delta}\psi$ exhibits a "finite speed of propagation" in the following quantitative sense:

$$\big\|\mathbb{1}_{(>tR)}\, e^{it\Delta}\psi\big\|_2 \leqslant \frac{C}{R^k} \qquad\qquad (\forall |t|\geqslant T_0)$$

where $\mathbb{1}_{(>tR)}$ is the characteristic function of the region $\{x\in\mathbb{R}^d\,|\,|x|\geqslant 4\pi t R\}$ and where the constant $C$ is independent of $R$ and depends only on $T_0$, $\psi$, and $k$.

**Exercise 12.** Let $\rho \in L^{p_1}(\mathbb{R}^3)\cap L^{p_2}(\mathbb{R}^3)$ with $1\leqslant p_1 < \frac{3}{2} < p_2 \leqslant \infty$.

(i) Prove that the $\mathbb{R}^3\to\mathbb{C}$ function

$$x \mapsto \frac{1}{|x|} * \rho$$

is a bounded continuous function vanishing as $|x|\to\infty$.

(ii) Using Young's inequality prove the bound

$$\Big\|\frac{1}{|\cdot|} * \rho\Big\|_\infty \leqslant C_{p_1,p_2}\,\|\rho\|_{p_1}^{\frac{\frac{2}{3}-\frac{1}{p_2}}{\frac{1}{p_1}-\frac{1}{p_2}}}\,\|\rho\|_{p_2}^{\frac{\frac{1}{p_1}-\frac{2}{3}}{\frac{1}{p_1}-\frac{1}{p_2}}} \qquad (*)$$

for some constant $C_{p_1,p_2}$ depending on $p_1$ and $p_2$ only (and blowing up as $p_1\to\frac{3}{2}$ or $p_2\to\frac{3}{2}$).

(iii) Let $p_1 = \frac{3}{2}-\varepsilon$ and $p_2 = \frac{3}{2}+\varepsilon$ with $\varepsilon>0$. Show that as $\varepsilon\to 0$ the product of norms in the r.h.s. of $(*)$ converges to $\|\rho\|_{3/2}$ (the constant $C_{p_1,p_2}$ obviously blowing up) but an inequality of the form $\|\,|\cdot|^{-1}*\rho\,\|_\infty \leqslant C\,\|\rho\|_{3/2}$ is false.