Functional Analysis

Institute of Mathematics, LMU Munich – Spring Term 2012
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PROBLEM IN CLASS – WEEK 12

These additional problems are for your own preparation at home. They supplement examples and properties not discussed in class. Some of them will discussed interactively in the weekly exercise/tutorial sessions. You are not required to hand in their solution. You are encouraged to think them over and to solve them. Being able to solve them is essential for the final exam. Further info at `www.math.lmu.de/~michel/SS12_FA.html`.

Problem 45. (L^2 as a first category set that is not nowhere dense)

For every $n \in \mathbb{N}$ let $M_n := \left\{ f \in L^2[0,1] \,\middle|\, \int_0^1 |f(x)|^2 \mathrm{d}x \leqslant n \right\}$.

(i) Show that $L^2[0,1] = \bigcup_{n=1}^\infty M_n$.

(ii) Show that each M_n is a closed subset in $L^1[0,1]$.

(iii) Show that the interior of each M_n, in the topology of $L^1[0,1]$, is empty.

(iv) From (i)–(iii) it appears that $L^2[0,1]$ is the countable union of nowhere dense sets. Why does not this contradict Baire's theorem?

Problem 46. (An application of the Uniform Boundedness Principle)

Let $\{a_n\}_{n=1}^\infty$ be a sequence of complex numbers such that

$$\sum_{n=1}^\infty a_n b_n < \infty \qquad \text{for all sequences } \{b_n\}_{n=1}^\infty \text{ in } c_0.$$

Prove that $\{a_n\}_{n=1}^\infty$ is in ℓ^1.

Problem 47. (An application of the Closed Graph theorem: projections on Banach spaces)

Let X_1, X_2 be two subspaces in the Banach space X such that $X_1 \cap X_2 = \emptyset$ and $\mathrm{Span}\,\{X_1, X_2\} = X$. In particular, this implies that every $x \in X$ can be uniquely written as $x = x_1 + x_2$ with $x_1 \in X_1$ and $x_n \in X_n$. Let $P : X \to X$ be the "projection onto X_1 along X_2", i.e., if $x = x_1 + x_2$ with $x_1 \in X_1$ and $x_2 \in X_2$ then $Px = x_1$. Prove that the operator P is bounded if and only if the subspaces X_1, X_2 are closed.

Problem 48. (Consequence of Hahn-Banach: unit ball as countable intersection of half-planes)

Show that the unit ball in $L^2_{\mathbb{R}}[0,1]$ (the real vector space of square-summable functions $[0,1] \to \mathbb{R}$) can be represented as the intersection of countably many half-spaces. (Recall that a half-space in a real normed space X is the set $\{x \in X \,|\, \phi(x) \leqslant 1\}$ for some $\phi \in X'$.)