Functional Analysis

Institute of Mathematics, LMU Munich – Spring Term 2012
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PROBLEM IN CLASS – WEEK 9

These additional problems are for your own preparation at home. They supplement examples and properties not discussed in class. Some of them will discussed interactively in the weekly exercise/tutorial sessions. You are not required to hand in their solution. You are encouraged to think them over and to solve them. Being able to solve them is essential for the final exam. Further info at `www.math.lmu.de/~michel/SS12_FA.html`.

Problem 33. (An orthogonal projection on $L^2[-1,1]$)

Consider the following subspaces $\mathcal{M}_1$ and $\mathcal{M}_2$ of the Hilbert space $\mathcal{H} = L^2[-1,1]$:

$$\begin{aligned} \mathcal{M}_1 &= \Big\{ f \in \mathcal{H} \,\Big|\, \int_{-1}^{1} f(t)\,\mathrm{d}t = 0 \Big\} \\ \mathcal{M}_2 &= \{ f \in \mathcal{H} \mid f(-t) = f(t) \ \text{ for a.e. } t \in [-1,1] \}\,. \end{aligned}$$

(i) Show that $\mathcal{M}_1$ is closed and find $\mathcal{M}_1^\perp$.

(ii) Show that $\mathcal{M}_2$ is closed and find $\mathcal{M}_2^\perp$.

(iii) Compute the distance from f to $\mathcal{M}_1$ with $f(x) = x^2$ and find the decomposition of f in $\mathcal{M}_1 \oplus \mathcal{M}_1^\perp$.

(iv) Compute the distance from f to $\mathcal{M}_2$ with $f(x) = e^x$ and find the decomposition of f in $\mathcal{M}_2 \oplus \mathcal{M}_2^\perp$.

Problem 34. (Examples of compact / non-compact subsets of $C([0,1])$)

Which of these subsets of $C([0,1])$ are pre-compact (i.e., their closure is compact) with respect to the $\|\ \|_\infty$-norm topology?

(i) $\{ f \mid f \in C^1([0,1])\,, f(0) = 0\,, \text{ and } |f'(x)| \leqslant 1 \,\forall x \in [0,1] \}$

(ii) $\{ f_n \mid f_n(x) = x^n \ \forall x \in [0,1]\,, n \in \mathbb{N} \}$ (This case is the remark after Def. 3.2 in class.)

(iii) $\{ f_n \mid f_n(x) = \sin nx \ \forall x \in [0,1]\,, n \in \mathbb{N} \}$

(iv) $\{ f_n \mid f_n(x) = n\, e^{-nx} \ \forall x \in [0,1]\,, n \in \mathbb{N} \}$

(v) $\{ f \mid f \in C^2([0,1])\,, f(0) = 0\,, \text{ and } |f''(x)| \leqslant 1 \,\forall x \in [0,1] \}$

(vi) $\{ f_\alpha \mid f_\alpha(x) = \sin(x + \alpha) \ \forall x \in [0,1]\,, \alpha \in \mathbb{R} \}$

Problem 35. (Examples of compact / non-compact operators on a Banach space)

(i) Let $T : X \to X$ be a compact operator on an infinite dimensional Banach space X. Does T have a bounded inverse? Justify your answer.

(ii) You know from class that the composition of a compact with a bounded operator is always compact. Produce instead an example of a Banach space X and a *non*-compact bounded $T : X \to X$ such that T^2 is compact.

Which of the following (bounded, linear) operators is compact? Justify your answer.

(iii) $T : C([0, 1]) \to C([0, 1])$, $Tf(x) := xf(x)\ \forall x \in [0, 1]\ \forall f \in C([0, 1])$

(iv) $T : C([0, 1]) \to C([0, 1])$, $Tf(x) := f(0) + xf(1)\ \forall x \in [0, 1]\ \forall f \in C([0, 1])$

(v) $T : C[0, 1] \to C[0, 1]$, $(Tf)(x) := \int_0^1 K(x, y)f(y)\,\mathrm{d}y\ \forall x \in [0, 1]\ \forall f \in C([0, 1])$, where $K : [0, 1] \times [0, 1] \to \mathbb{C}$ is continuous

(vi) $T : C[0, 1] \to C[0, 1]$, $(Tf)(x) := \int_0^1 e^{tx} f(t)\,\mathrm{d}t\ \forall x \in [0, 1]\ \forall f \in C([0, 1])$

(vii) $T : \ell^p \to \ell^p$, $p \in [1, \infty]$, $Tx := (0, x_1, x_2, \dots)\ \forall x = (x_1, x_2, \dots) \in \ell^p$.

Problem 36. (Inclusion of spaces reverts for their dual if one space is dense in the other. Variational characterisation of the norm of an operator on a Hilbert space.)

Recall that $A \hookrightarrow B$, where A and B are normed spaces, denotes a continuous embedding of A into B, that is, an injective bounded linear map from A to B.

(i) Give two examples of normed vector spaces X and Y such that $X \subset Y$ but $X' \not\hookleftarrow Y'$. (Thus, the implication $X \hookrightarrow Y \Rightarrow X' \hookleftarrow Y'$ is tempting but false in general.)

(ii) Let Y be a normed vector spaces and X a subspace of Y *dense* in Y. Show that $X' \hookleftarrow Y'$. Give an example of such a situation.

(iii) Let $\mathcal{H}$ be a Hilbert space, $\mathscr{D}$ be a dense subspace in $\mathcal{H}$ and T be a bounded linear operator on $\mathcal{H}$. Show that

$$\|T\| = \sup_{\substack{x,y \in \mathcal{H} \\ \|x\|=\|y\|=1}} \big|\langle x, Ty\rangle\big| = \sup_{\substack{x,y \in \mathscr{D} \\ \|x\|=\|y\|=1}} \big|\langle x, Ty\rangle\big|\,.$$