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PROBLEM IN CLASS – WEEK 8

These additional problems are for your own preparation at home. They supplement examples and properties not discussed in class. Some of them will discussed interactively in the weekly exercise/tutorial sessions. You are not required to hand in their solution. You are encouraged to think them over and to solve them. Being able to solve them is essential for the final exam. Further info at `www.math.lmu.de/~michel/SS12_FA.html`.

Problem 29. (Projections onto closed convex sets)

Let $\mathcal{H}$ be a Hilbert space. Let Σ be a closed and *convex* non-empty subset of $\mathcal{H}$.

(i) Show that there exists a unique element $P_{\Sigma}(x) \in \Sigma$ such that $\mathrm{dist}(x, \Sigma) = \|x - P_{\Sigma}(x)\|$.

(ii) Show that if Σ is a non-trivial linear subspace of $\mathcal{H}$ then $P_{\Sigma}(x)$ is precisely the orthogonal projection of x onto Σ given by the Projection Theorem.

(iii) Show that, for any $x, z \in \mathcal{H}$ and any scalar λ,

$$
\begin{aligned}
P_{z+\Sigma}(x) &= z + P_{\Sigma}(x - z)\,, \\
P_{\lambda\Sigma}(\lambda x) &= \lambda P_{\Sigma}(x)\,.
\end{aligned}
$$

(iv) Show that if $x \notin \Sigma$ then $P_{\Sigma}(x) \in \partial\Sigma$ and $\mathrm{dist}(x, \Sigma) = \mathrm{dist}(x, \partial\Sigma)$.

(v) Assume further that $\partial\Sigma$ is convex. Show that if $x \notin \Sigma$ then $P_{\Sigma}(x) = P_{\partial\Sigma}(x)$.

Problem 30. (Projection onto a sub-graph of a convex function. Projection onto a closed ball, a hyperplane, a half-plane.)

(i) Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 convex function and let $\Sigma := \{(x, y) \in \mathbb{R}^2 \,|\, f(x) \leqslant y\}$. Let $P_{\Sigma}(a, b)$ be the projection of (a, b) onto Σ, as defined in Problem 29(i). Show that if $(a, b) \notin \Sigma$ then $P_{\Sigma}(a, b) = (x, f(x))$ where x is the solution to the equation $(b - f(x))f'(x) + a - x = 0$.

(ii) Let $\Sigma := \{(x, y) \in \mathbb{R}^2 \,|\, x^2 \leqslant y\}$. Find the point $P_{\Sigma}(1, \frac{1}{2})$ in $\mathbb{R}^2$ where $P_{\Sigma} : \mathbb{R}^2 \to \mathbb{R}^2$ is the orthogonal projection operator onto Σ.

Let now $\mathcal{H}$ be a Hilbert space on $\mathbb{R}$ or $\mathbb{C}$, $x \in \mathcal{H}$, Σ be the closed and convex non-trivial subset of $\mathcal{H}$ specified below, and $P_{\Sigma}(x)$ be the projection of x onto Σ. Find $P_{\Sigma}(x)$ when

(iii) $\Sigma =$ the closed ball of $\mathcal{H}$ with radius $R > 0$ centred at $x_0 \in \mathcal{H}$,

(iv) $\Sigma = \{x \in \mathcal{H} \,|\, \langle z, x\rangle = a\}$ for a given $z \in \mathcal{H} \setminus \{0\}$ and a given scalar a,

(v) (assuming further that $\mathcal{H}$ is a real Hilbert space) $\Sigma = \{x \in \mathcal{H} \,|\, \langle z, x\rangle \leqslant a\}$ for a given $z \in \mathcal{H} \setminus \{0\}$ and a given $a \in \mathbb{R}$.

Problem 31. (Existence of algebraic basis. Non-countable bases.)

(i) Let V be a vector space. Show, using Zorn's Lemma, that V has an algebraic basis.

(ii) Is $\{e_n\}_{n=1}^{\infty}$, $e_1 = (1, 0, 0, 0, \dots)$, $e_2 = (0, 1, 0, 0, \dots)$, $e_3 = (0, 0, 1, 0, \dots)$, $\dots$, an algebraic basis of ℓ^∞? Justify your answer.

(iii) Consider the vector space $\ell^\infty(\mathbb{Z}_2) = \{a = (a_1, a_2, \dots) \,|\, a_n \in \mathbb{Z}_2\}$ where $\mathbb{Z}_2$ is the finite field of two elements (i.e., it has only two elements, 0 and 1, and the arithmetic is done modulo 2, i.e., $0+1 = 1$, $1+1 = 0$, etc.) $\mathbb{Z}_2$ has a natural vector space structure over the field $\mathbb{Z}_2$. Show that $\ell^\infty(\mathbb{Z}_2)$ does not have a countable algebraic basis.

Problem 32. (Existence of unbounded linear functionals. They have dense kernel.)

(i) Let $(X, \| \ \|)$ be an infinite-dimensional normed space. Show that there exists a non-continuous linear functional on X.

(ii) Let ϕ be a non-continuous linear functional on a normed space $(X, \| \ \|)$. Show that $\operatorname{Ker} \phi$ is dense in X.