Institute of Mathematics, LMU Munich – Spring Term 2012
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**PROBLEM IN CLASS – WEEK 6**

*These additional problems are for your own preparation at home. They supplement examples and properties not discussed in class. Some of them will discussed interactively in the weekly exercise/tutorial sessions. You are not required to hand in their solution. You are encouraged to think them over and to solve them. Being able to solve them is essential for the final exam. Further info at* `www.math.lmu.de/~michel/SS12_FA.html`.

**Problem 21.** (Computation of operator norms.)

Consider the Banach space $C([0,1])$ of (real- or complex-valued) continuous functions on $[0,1]$, with its natural norm $\|f\| = \max_{x\in[0,1]}|f(x)|$, and the linear operator $T : C([0,1]) \to C([0,1])$ defined in each case below for any $f \in C([0,1])$ and $x \in [0,1]$. Compute the operator norm $\|T\|$.

(i) $(Tf)(x) := \displaystyle\int_0^x f(y)\,\mathrm{d}y$.

(ii) $(Tf)(x) := x^2 f(0)$.

(iii) $(Tf)(x) := f(x^2)$.

(iv) $T^n$, $n \in \mathbb{N}$, where $T$ is defined as in (i).

**Problem 22.** (The value attained by a functional at a point is determined by distance from kernel. Characterization of when a functional attains its norm.)

Let $(X, \|\ \|_X)$ be a normed space and let $\phi$ be a bounded linear functional on $X$.

(i) Show that $|\phi(x)| = \|\phi\|_{X'} \mathrm{dist}(x, \mathrm{Ker}\,\phi)$ $\forall x \in X$.

(ii) Show that $|\phi(x) - a| = \|\phi\|_{X'} \mathrm{dist}(x, \phi^{-1}(\{a\}))$ $\forall x \in X$, $\forall a \in \mathbb{K}$, assuming $\phi \neq 0$.

(iii) Show that for any non-zero $\phi$ the following are equivalent:

  (a) $\phi$ does not attain its norm (i.e., the supremum in $\|\phi\|_{X'} = \sup\limits_{\substack{x\in X\\ \|x\|_X \leqslant 1}} |\phi(x)|$ is not attained).

  (b) There is no $x \in X$ such that $\|x\| = 1$ and $\mathrm{dist}(x, \mathrm{Ker}\,\phi) = 1$.

  (c) The distance from any $x \in X \setminus \mathrm{Ker}\,\phi$ to $\mathrm{Ker}\,\phi$ is never attained, i.e., $\mathrm{dist}(x, \mathrm{Ker}\,\phi) < \|x - x_0\|_X$ $\forall x_0 \in \mathrm{Ker}\,\phi$.

(iv) Can (b) above occur or would that contradict Riesz' lemma? Explain your answer.

**Problem 23.** (Norm-distance between hyperplanes.)

(i) Let $(X, \| \ \|)$ be a normed space, let $\phi$ be a non-zero bounded linear functional on $X$, and let $\Pi_1$ and $\Pi_2$ be the hyperplanes $\Pi_j := \{x \in X \,|\, \phi(x) = a_j\}$ for some fixed $a_j \in \mathbb{K}$, $j = 1, 2$. Show that

$$\mathrm{dist}(\Pi_1, \Pi_2) \;=\; \frac{|a_1 - a_2|}{\|\phi\|_{X'}}\,.$$

(ii) Deduce from (i) the well-known formula

$$\mathrm{dist}(\Pi_1, \Pi_2) \;=\; \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

that gives the distance from the two parallel planes $\Pi_1$ and $\Pi_2$ in $\mathbb{R}^3$ defined by the equations $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$, $(x, y, z) \in \mathbb{R}^3$.

(iii) Compute the distance in $C([0, 1], \| \ \|_\infty)$ between the two subsets

$$E_1 := \Big\{ f \in C([0,1]) \,\Big|\, \int_0^1 f(x)\,\mathrm{d}x = 1 \Big\} \qquad \text{and} \qquad E_2 := \Big\{ f \in C([0,1]) \,\Big|\, \int_0^1 f(x)\,\mathrm{d}x = -1 \Big\}\,.$$

**Problem 24.** Dual of $c_0$. Dual of $\ell^1$. Dual of $\ell^\infty$.

(i) Show that $(c_0)' \cong \ell^1$ via the duality $x \overset{y \in \ell^1}{\longmapsto} \displaystyle\sum_{n=1}^\infty y_n x_n$.

(This means: the map $y \mapsto \phi_y$, $\phi_y(x) := \displaystyle\sum_{n=1}^\infty y_n x_n$, defines an isometric isomorphism of $\ell^1$ onto $(c_0)'$, in the sense of isomorphism of normed spaces, i.e., $\phi$ is a surjective linear isometry that preserves linear and normed structure.)

(ii) Show that $(\ell^1)' \cong \ell^\infty$ via the duality $x \overset{y \in \ell^\infty}{\longmapsto} \displaystyle\sum_{n=1}^\infty y_n x_n$.

(iii) Show that $\ell^1$ embeds isometrically into $(\ell^\infty)'$. Show in fact that $(\ell^\infty)' \cong c_0^\perp \oplus \ell^1$, where $c_0^\perp := \{\eta \in (\ell^\infty)' \,|\, \eta(x) = 0 \,\forall x \in c_0\}$.
Note: after the Hahn-Banach theorem will be introduced, we shall see that in fact $c_0^\perp \neq 0$, that is to say, $\ell^1$ embeds isometrically into $(\ell^\infty)'$ but *not* onto.