Institute of Mathematics, LMU Munich – Spring Term 2012
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PROBLEM IN CLASS – WEEK 4

These additional problems are for your own preparation at home. They supplement examples and properties not discussed in class. Some of them will discussed interactively in the weekly exercise/tutorial sessions. You are not required to hand in their solution. You are encouraged to think them over and to solve them. Being able to solve them is essential for the final exam. Further info at `www.math.lmu.de/~michel/SS12_FA.html`.

Problem 13. (Cauchy and fast Cauchy. A metric is uniformly continuous in each entry. Continuous functions between metric spaces are uniformly continuous if the domain is compact.)

Let (X, d), (Y, ρ) be metric spaces. Recall that $f : X \to Y$ is UNIFORMLY CONTINUOUS if $\forall \varepsilon > 0\ \exists\, \delta > 0$ such that $\forall x, x' \in X \;\; d(x, x') < \delta \Rightarrow \rho(f(x), f(x') < \varepsilon$.

(i) Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in X. Assume that there is a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ that converges to $x \in X$. Show that $x_n \xrightarrow{n\to\infty} x$.

(ii) A sequence $\{x_n\}_{n=1}^{\infty}$ in X is called FAST CAUCHY SEQUENCE if $\displaystyle\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$. Show that a fast Cauchy sequence is a Cauchy sequence.

(iii) Show that every Cauchy sequence in X has a subsequence that is a fast Cauchy sequence.

(iv) Fix $x_0 \in X$, arbitrary. Show that $x \mapsto d(x, x_0)$ is a uniformly continuous function from X to $\mathbb{R}$.

(Note: in class, Remark 1.29(4), it was proved that $d : X \times X \to \mathbb{R}$ is continuous.)

(v) Assume in addition that X is compact (as a topological space with the metric topology). Show that every continuous function $f : X \to Y$ is necessarily uniformly continuous.

Problem 14. (Subspaces of separable metric spaces are separable. Lindelöf's theorem. For metric spaces compact = sequentially compact.)

(i) Show that a subspace of a separable metric space is separable.

(ii) (Lindelöf's theorem.) Let X be a second countable metric space. Show that every open cover of X has a countable subcover.

(iii) Show that a metric space is compact if and only if it is sequentially compact.

(iv) Show that a sequentially compact metric space is separable.

Problem 15. (The completion theorem in full detail. Uniqueness of the completion up to isometry.)

Let (X,d) be a metric space. Let $\mathcal{C}_X$ be the set of Cauchy sequences in X. Define a relation $\sim$ in $\mathcal{C}_X$ by declaring "$\{s_n\}_{n=1}^\infty \sim \{t_n\}_{n=1}^\infty$" to mean that $d(s_n,t_n)\to 0$ as $n\to\infty$.

(i) Show that the relation "$\sim$" is an equivalence relation.

(ii) Let $\widetilde{X}$ denote the set of equivalence classes of $\mathcal{C}_X$ and let $\widetilde{s}$ denote the equivalence class of $\langle s\rangle \equiv \{s_n\}_{n=1}^\infty$. Show that the function $\widetilde{d}:\widetilde{X}\times\widetilde{X}\to[0,\infty)$ defined by

$$\widetilde{d}(\widetilde{s},\widetilde{t}\,) := \lim_{n\to\infty} d(s_n,t_n) \qquad \widetilde{s},\widetilde{t}\in\widetilde{X}$$

is well-defined (i.e., it does not depend on the choice of the representative in each equivalence class) and is a metric on $\widetilde{X}$.

(iii) Show that $(\widetilde{X},\widetilde{d})$ is complete.

(iv) For $x\in X$ define $\widetilde{x}$ to be the equivalence class of the constant sequence $\{x,x,x,\dots\}$. Show that the function $x\mapsto\widetilde{x}$ is an isometry of (X,d) onto a dense subset of $(\widetilde{X},\widetilde{d})$.

So far: any (X,d) is isometrically embedded in a dense of a complete $(\widetilde{X},\widetilde{d})$ (in the above concrete construction this is done by identifying each $x\in X$ with the equivalence class $\widetilde{x}$ of the constant sequence $\{x,x,x,\dots\}$). Such $(\widetilde{X},\widetilde{d})$, together with the isometric embedding $\phi: X\to\phi(X)\subset\widetilde{X}$, is called COMPLETION of (X,d).

(v) Assume that $(\widetilde{X}_1,\widetilde{d}_1,\phi_1)$ and $(\widetilde{X}_2,\widetilde{d}_2,\phi_2)$ are two completions of (X,d). Show that $\phi_2\circ\phi_1^{-1}$ maps isometrically $\phi_1(X)$ onto $\phi_2(X)$ and extends to an isometry $\widetilde{X}_1\to\widetilde{X}_2$.

(In other words: the completion of a metric space is unique, up to isometry.)

Problem 16. (Contractions with no fixed points. Compact implies bounded. Product of compacts is compact. Compactness = finite intersection property.)

(i) Produce a (non-complete) metric space (X,d) and a contraction $\Phi: X\to X$ such that Φ has no fixed points.

(ii) Show that every compact metric space is bounded.

(iii) Show that the product of two compact topological spaces is compact in the product topology.

(iv) Show that a topological space $(X,\mathcal{T})$ is compact if and only if any family $\mathscr{F}$ of closed subsets of X with $\bigcap_{n=1}^{N} F_n\neq\emptyset$ for any finite subfamily $\{F_n\}_{n=1}^N$ in $\mathscr{F}$ satisfies $\bigcap_{F\in\mathscr{F}} F_n\neq\emptyset$.
(A topological space where any family $\mathscr{F}$ of closed subsets has the feature above is said to have the FINITE INTERSECTION PROPERTY.)