Functional Analysis

Institute of Mathematics, LMU Munich – Spring Term 2012
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PROBLEM IN CLASS – WEEK 2

These additional problems are for your own preparation at home. They supplement examples and properties not discussed in class. Some of them will discussed interactively in the weekly exercise/tutorial sessions. You are not required to hand in their solution. You are encouraged to think them over and to solve them. Being able to solve them is essential for the final exam. Further info at `www.math.lmu.de/~michel/SS12_FA.html`.

Problem 5. (Examples of closure, interior, boundary, limit/isolated points in $\mathbb{R}$. Examples of simultaneously closed and open sets.)

(i) Consider the following subsets of $\mathbb{R}$:

$$A = [0,1), \qquad B = \mathbb{Q}, \qquad C = \mathbb{R}\backslash\mathbb{Q}, \qquad D = \left\{\frac{1}{n} \,|\, n \in \mathbb{N}\right\} \cup (2,3]\,.$$

Find for each of them: closure, interior, boundary, limit points, isolated points with respect to the ordinary metric topology in $\mathbb{R}$.

(ii) Consider the topological space $X = [0,1] \cup [2,3]$ equipped with the usual metric topology induced by $\mathbb{R}$. Produce a non-trivial subset of X which is both open and closed.

(iii) Equip $\mathbb{Q}$ with the usual metric topology induced by $\mathbb{R}$ and produce a non-trivial subset of the topological spaces thus obtained which is both open and closed.

Problem 6. (The HALF-OPEN INTERVAL TOPOLOGY.)

Consider the set $\mathbb{R}$ equipped with the topology $\mathcal{T}$ generated by the base $\mathscr{B}$ consisting of $\emptyset$ and of all subsets of $\mathbb{R}$ of the form $[a,b)$, $a,b \in \mathbb{R}$, $a < b$. (The fact that $\mathscr{B}$ is a base for a topology $\mathcal{T}$ in $\mathbb{R}$ is proved in Exercise 6(iii).)

(i) Show that every subset of $\mathbb{R}$ which is open in the usual metric topology of $\mathbb{R}$ is also $\mathcal{T}$-open.

(ii) Show that each interval $[a,b)$ (which is $\mathcal{T}$-open, by assumption) is $\mathcal{T}$-closed.

(iii) Let $E \subset \mathbb{R}$ and $x \in \mathbb{R}$. Show that $x \in \overline{E}$ (the $\mathcal{T}$-closure of E) if and only if there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of points in E such that $x_n \geqslant x$ and $|x_n - x| \xrightarrow{n\to\infty} 0$.

(iv) Show that a function f from the topological space $(\mathbb{R},\mathcal{T})$ to the topological space $(\mathbb{R},\mathcal{T}_{\mathrm{metric}})$ (that is, the reals with the usual metric topology) is continuous *if and only if* it is continuous from the right in the ordinary sense, that is,

$$\lim_{\varepsilon\downarrow 0} f(x+\varepsilon) = f(x) \qquad \forall x \in \mathbb{R}. \tag{$\bullet$}$$

(Note that $\varepsilon > 0$ in ($\bullet$).)

Problem 7. (Either-or topology. Partition topology. Sub-base of straight lines in the plane.)

(i) Consider the family $\mathcal{T}$ of subsets of $X = [-1, 1]$ which either do not contain $\{0\}$ or contain $(-1, 1)$. Show that $\mathcal{T}$ is a topology in X (the EITHER-OR TOPOLOGY) and find all $\mathcal{T}$-closed sets in $(X, \mathcal{T})$.

(ii) Consider the family $\mathscr{B}$ of subsets of a given set X such that: $\emptyset \in \mathscr{B}$, all elements in $\mathscr{B}$ are mutually disjoint, the union of all elements in $\mathscr{B}$ is the whole X. Show that $\mathscr{B}$ is a base for a topology $\mathcal{T}_{\text{part}}$ in X (the PARTITION TOPOLOGY) and show that every open set in $(X, \mathcal{T}_{\text{part}})$ is also closed.

(iii) Consider in the Euclidean plane $\mathcal{E}$ the family $\mathscr{B}$ consisting of any finite intersection of straight lines. Show that $\mathscr{B}$ is a base for a topology in $\mathcal{E}$. What is this topology?

Problem 8. (Kuratowski's closure-complement problem.)

(i) Denote for short $A^- \equiv \overline{A}$ and $A' \equiv \mathbb{R} \backslash A$ for every $A \subset \mathbb{R}$ with respect to to the ordinary metric topology. Consider the set

$$E = (0, 1) \cup (1, 2) \cup \{3\} \cup \big(\mathbb{Q} \cap [4, 5]\big)$$

Find the following 14 sets:

(closure-complement chain) E, E^-, $E^{-\prime}$, $E^{-\prime-}$, $E^{-\prime-\prime}$, $E^{-\prime-\prime-}$, $E^{-\prime-\prime-\prime}$,
(complement-closure chain) E', $E^{\prime-}$, $E^{\prime-\prime}$, $E^{\prime-\prime-}$, $E^{\prime-\prime-\prime}$, $E^{\prime-\prime-\prime-}$, $E^{\prime-\prime-\prime-\prime}$.

Check by inspection that they are all distinct and that proceeding further along either chain no new distinct sets are formed.

(ii) Let X be a topological space and $E \subset X$. Show that by repeatedly applying to E the operations of closure and complement in alternate order one can obtain *at most* 14 distinct sets, including E.

(*Hint:* a good strategy is to show that $E^{-\prime-\prime-\prime-} = E^{-\prime-}$ and $E^{\prime-\prime-\prime-\prime-} = E^{\prime-\prime-}$, so that either chain considered in (i) closes without giving further distinct sets. Deduce such two identities from

(a) E open $\Rightarrow E^- = E^{-i-}$
(b) E closed $\Rightarrow E^i = E^{i-i}$
(c) $E^{-\prime} = E^{\prime i}$
(d) $E^{\prime-} = E^{i\prime}$

where $A^i \equiv \mathring{A}$, the interior of A. Then one has only to prove (a)-(d).)

(Reference: This result was first published by K. Kuratowski in *Sur l'opération $\overline{A}$ de l'Analysis Situs*, Fund. Mathem. 3 (1922) 182-199. Note: "analysis situs" is the old-fashioned name for "topology" – the Latin etymology of the former and the Greek etymology of the latter are the same.)