Functional Analysis

Institute of Mathematics, LMU Munich – Spring Term 2012
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HOMEWORK ASSIGNMENT no. 11, issued on Tuesday 26 June 2012
Due: Tuesday 3 July 2012 by 6 pm in the designated "FA" box on the 1st floor
Info: `www.math.lmu.de/~michel/SS12_FA.html`

Each exercise is worth a full mark of 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in your solutions either in German or in English.

Exercise 41. (Integral operators on $C[0,1]$ and $L^p[0,1]$. Boundedness and compactness. The Hilbert-Schmidt norm attains the operator norm when the kernel factorizes.)

(i) For any $f \in C[0,1]$ define $(Tf)(x) := \displaystyle\int_0^1 \frac{f(y)}{|x-y|^{1/3}}\,\mathrm{d}y$, $x \in [0,1]$. Show that T is a bounded and compact linear map from $C([0,1])$ into itself.

(ii) Let $K : [0,1] \times [0,1] \to \mathbb{C}$ be a continuous function. Consider the integral operator $f \overset{T}{\longmapsto} Tf$ given by $(Tf)(x) = \int_0^1 K(x,y)\, f(y)\,\mathrm{d}y$ and denote by $\|T\|_{p\to q}$ its norm as a $L^p[0,1] \to L^q[0,1]$ map. Show that

- $\|T\|_{1\to 1} \;\leqslant\; \sup_{y\in\mathbb{R}} \displaystyle\int_0^1 |K(x,y)|\,\mathrm{d}x$
- $\|T\|_{2\to 2} \;\leqslant\; \Big(\displaystyle\int_{[0,1]^2} |K(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y \Big)^{1/2}$
- $\|T\|_{\infty\to\infty} \;\leqslant\; \sup_{x\in\mathbb{R}} \displaystyle\int_0^1 |K(x,y)|\,\mathrm{d}y\,.$

(iii) Consider the case $p = q = 2$ of part (ii). Show that $\|T\|_{2\to 2} = \|K\|_{L^2([0,1]^2)}$ if and only if $\exists\, K_1, K_2 \in L^2[0,1]$ such that $K(x,y) = K_1(x)K_2(y)$ for a.e. $x, y \in [0,1]$.

Exercise 42. (Ehrling's lemma. L^1-norm and derivative's sup control the sup norm. An Ehrling-like interpolation on gradients.)

(i) Let X, Y, and Z be Banach spaces with norms $\|\ \|_X$, $\|\ \|_Y$, and $\|\ \|_Z$, respectively. Assume that $X \subset Y$ with compact injection and $Y \subset Z$ with continuous injection. (I.e., $\mathrm{id} : X \to Y$ is compact and $\mathrm{id} : Y \to Z$ is bounded.) Prove that

$$\forall \varepsilon > 0 \;\; \exists\, C_\varepsilon \geqslant 0 \;\text{ such that }\; \|x\|_Y \leqslant \varepsilon\,\|x\|_X + C_\varepsilon \|x\|_Z \quad \text{for all } x \in X.$$

(ii) Show that $\forall \varepsilon > 0 \;\; \exists\, C_\varepsilon \geqslant 0$ such that

$$\max_{x\in[0,1]} |u(x)| \;\leqslant\; \varepsilon \max_{x\in[0,1]} |u'(x)| \;+\; C_\varepsilon\,\|u\|_{L^1[0,1]} \qquad \forall\, u \in C^1([0,1])\,.$$

(iii) Let $d \in \mathbb{N}$, $R > 0$ and $\Omega := B_R(0)$ in $\mathbb{R}^d$. Show that $\forall \varepsilon > 0 \;\; \exists\, C_\varepsilon \geqslant 0$ such that

$$\|\nabla u\|_{C^0(\overline{\Omega})} \;\leqslant\; \varepsilon\,\|D^2 u\|_{C^0(\overline{\Omega})} + C_\varepsilon \|u\|_{C^0(\overline{\Omega})} \qquad \forall\, u \in C^2(\overline{\Omega})$$

and give an explicit estimate of the constant C_ε.

Exercise 43. (Applications of Fatou's Lemma and of monotone convergence. Boundedness of the L^p-L^q multiplication operator.)

(i) Let (X, μ) be a measure space and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of integrable functions on (X, μ). Suppose that there exists an integrable function f such that

- $f_n \to f$ pointwise almost everywhere,
- $\int_X |f_n| \, \mathrm{d}\mu \to \int_X |f| \, \mathrm{d}\mu$ as $n \to \infty$.

Show that $\int_X |f - f_n| \, \mathrm{d}\mu \to 0$ as $n \to \infty$.

(ii) Let $a : [0,1] \to \mathbb{C}$ be a measurable function. Let $T_a : L^p[0,1] \to L^q[0,1]$, with $p, q \in [1, \infty]$, be the operator of pointwise multiplication by a, i.e., $(T_a f)(x) := a(x) f(x)$ for a.a. $x \in [0,1]$. Find the necessary and sufficient condition on a such that T_a is continuous

- when $p < q$,
- when $p \geqslant q$.

Exercise 44. (Orthonormal bases on an interval)

In this exercise you should use the fact that $\left\{ \frac{e^{inx}}{\sqrt{2\pi}} \right\}_{n \in \mathbb{Z}}$ is an orthonormal basis of $L^2[0, 2\pi]$, which was proved in Exercise 38(ii).

(i) Consider the collection $\{e_n\}_{n \in \mathbb{Z}}$ in $L^2[a,b]$ with $e_n(x) = e^{2\pi i n x}$. Prove that the orthogonal complement of such a collection

- is only $\{0\}$ if $b - a \leqslant 1$
- is different from $\{0\}$ if $b - a > 1$.

(ii) Show that $\{f_n\}_{n=0}^{\infty}$, where $f_n(x) := \sqrt{2} \cos \frac{\pi x}{2} (2n+1)$, is an orthonormal basis of $L^2[0,1]$.