Institute of Mathematics, LMU Munich – Spring Term 2012
Prof. T. Ø. Sørensen Ph.D, A. Michelangeli Ph.D


**HOMEWORK ASSIGNMENT no. 9**, issued on Tuesday 12 June 2012
**Due:** Tuesday 19 June 2012 by 6 pm in the designated "FA" box on the 1st floor
**Info:** `www.math.lmu.de/~michel/SS12_FA.html`

*Each exercise is worth a full mark of 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in your solutions either in German or in English.*

**Exercise 33.** (Every separable Banach space is a quotient of $\ell^1$)

Let $X$ be a separable Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Consider $\ell^1$ over the same field $\mathbb{K}$.

(i) Produce a bounded linear map from $\ell^1$ onto $X$ ("onto" = surjective).

(ii) Show that there exists a closed linear subspace $Y \subset \ell^1$ such that $X \cong \ell^1/Y$, where $\cong$ is a linear isometric isomorphism. (The quotient Banach space was introduced in Problem 28).

**Exercise 34.** (The non-separable Hilbert space of Besicovitch quasi-periodic functions)

Let $X := \left\{ f : \mathbb{R} \to \mathbb{C} \text{ of the form } f(x) = \sum_{k=1}^{n} c_k\, e^{i\alpha_k x} \,\middle|\, n \in \mathbb{N}\,, c_k \in \mathbb{C}\,, \begin{array}{c}\alpha_k \in \mathbb{R}\\ \text{all distinct}\end{array} \right\}$, equipped with the natural structure of complex vector space.

(i) Show that $\langle g, f\rangle := \lim_{R\to\infty} \frac{1}{2R} \int_{-R}^{R} \overline{g(x)}\, f(x)\, \mathrm{d}x$, $f, g \in X$, defines a scalar product $\langle\ ,\ \rangle$ on $X$.

(ii) Show that $X$ is non complete.

(iii) Show that the completion of $X$ is a non-separable Hilbert space.

**Exercise 35.** (Quadratically close orthonormal bases)

Let $\{\phi_n\}_{n=1}^{\infty}$ be an orthonormal basis of a Hilbert space $\mathcal{H}$ and let $\{\psi_n\}_{n=1}^{\infty}$ be an orthonormal system. Prove that if

$$\sum_{n=1}^{\infty} \|\phi_n - \psi_n\|^2 \;<\; \infty$$

then $\{\psi_n\}_{n=1}^{\infty}$ too is an orthonormal basis of $\mathcal{H}$.

**Exercise 36.** (Schur's test. Hilbert's matrix. Hankel's matrix.)

Let $\mathcal{H}$ be a separable Hilbert space and let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of $\mathcal{H}$.

(i) Let $T : \mathrm{Span}(\{e_n\}_{n=1}^{\infty}) \to \mathcal{H}$ be a linear operator such that

$$\sum_{n=1}^{\infty} b_n |\langle e_m, Te_n \rangle| \leqslant A\, a_m \ \ \forall m \in \mathbb{N} \qquad \text{and} \qquad \sum_{m=1}^{\infty} |\langle e_m, Te_n \rangle| a_m \leqslant B\, b_n \ \ \forall n \in \mathbb{N}$$

for some $A, a_1, a_2, a_3, \cdots > 0$ and some $B, b_1, b_2, b_3, \cdots > 0$. Show that $T$ extend uniquely to a bounded linear operator over the whole $\mathcal{H}$ with $\|T\| \leqslant \sqrt{AB}$.

(ii) Let $T : \mathrm{Span}(\{e_n\}_{n=1}^{\infty}) \to \mathcal{H}$ be a linear operator such that $\langle e_m, Te_n \rangle = (n + m - 1)^{-1}$ $\forall n, m \in \mathbb{N}$. Show that $T$ extend uniquely to a bounded linear operator over the whole $\mathcal{H}$ with $\|T\| \leqslant \pi$.

(iii) Let $T : \mathrm{Span}(\{e_n\}_{n=1}^{\infty}) \to \mathcal{H}$ be a linear operator such that $\langle e_m, Te_n \rangle = \dfrac{1}{2^{n+m-1}}$ $\forall n, m \in \mathbb{N}$. Show that $T$ extend uniquely to a bounded linear operator over the whole $\mathcal{H}$ and compute $\|T\|$.