Functional Analysis

Institute of Mathematics, LMU Munich – Spring Term 2012
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HOMEWORK ASSIGNMENT no. 8, issued on Tuesday 5 June 2012
Due: Tuesday 12 June 2012 by 6 pm in the designated "FA" box on the 1st floor
Info: `www.math.lmu.de/~michel/SS12_FA.html`

Each exercise is worth a full mark of 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in your solutions either in German or in English.

Exercise 29. (Another Hardy's operator on ℓ^p.)

Let $p \in (1, \infty)$ and consider the linear operator H defined by

$$(Hx)_n := \sum_{k=n}^{\infty} \frac{x_k}{k} \qquad (n \in \mathbb{N})$$

$\forall x = (x_1, x_2, \dots) \in c_{00} \subset \ell^p$. Show that H extends uniquely to a bounded linear operator $H : \ell^p \to \ell^p$ and compute its norm $\|H\|$.

Exercise 30. (A normed space has a inner product iff the parallelogram law holds true.)

(i) Let $(X, \| \; \|)$ be a normed space (on $\mathbb{R}$ or on $\mathbb{C}$, you are supposed to consider both cases.) Assume that the parallelogram law

$$\|x + y\|^2 + \|x - y\|^2 = 2\,\|x\|^2 + 2\,\|y\|^2$$

holds for all $x, y \in X$. Show that X is an inner product space, i.e., one can find an inner product $\langle \, , \rangle$ on X such that $\|x\|^2 = \langle x, x \rangle$.

(ii) For which $p \in [1, \infty]$ is ℓ^p a Hilbert space? Justify your answer.

Exercise 31. ($C([0,1])$ embeds isometrically into ℓ^∞, not into ℓ^p when p is finite.)

(i) Let $p \in (1, \infty)$. Let $x_1, x_2 \in \ell^p$ with $\|x_1\|_p = \|x_2\|_p = 1$. Show that

$$\left\| \frac{x_1 + x_2}{2} \right\|_p = 1 \quad \Rightarrow \quad x_1 = x_2 .$$

(*Hint:* consider when Minkowski's inequality in ℓ^p becomes an equality, Problem 25(i).)

(ii) For which $p \in (1, \infty]$ can the space $C([0,1])$ (with the usual supremum norm) be embedded isometrically into ℓ^p? Justify your answer and, in the affirmative cases, provide an explicit isometric embedding.

Exercise 32. (c and c_0 are not isometrically isomorphic, but their duals are.)

Consider the Banach spaces (on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$)

$$\begin{aligned} c &= \big\{ x = (x_1, x_2, x_3, \dots) \mid x_n \in \mathbb{K} \ \ \forall n \in \mathbb{N} \ \text{ and } \exists \lim_{n\to\infty} x_n \big\} \\ c_0 &= \big\{ x = (x_1, x_2, x_3, \dots) \mid x_n \in \mathbb{K} \ \ \forall n \in \mathbb{N} \ \text{ and } \lim_{n\to\infty} x_n = 0 \big\} \ \subset \ c \end{aligned}$$

with the standard norm $\|x\|_\infty = \sup_n |x_n|$.

(i) Prove that c_0 and c are not isometrically isomorphic.

(*Hint*: consider the closed unit balls in c_0 and in c and exploit the fact that one admits extremal points whereas the other does not, which is not compatible with the existence of a linear isometric bijection between c_0 and c. A point x in a convex set K of a normed space is called an *extremal point* if one cannot represent x as a non-trivial convex combination $x = tx_1 + (1-t)x_2$ where $t \in (0,1)$, $x_1, x_2 \in K$, $x_1 \neq x_2$.)

(ii) Prove that $c' \cong c_0' \cong \ell^1$.

(*Hint*: consider the linear functional $\phi_{\lim}$ on c defined by $\phi_{\lim}(x) := \lim_{n\to\infty} x_n$, $x = (x_1, x_2, \dots) \in c$, and prove that $c' \cong \phi_{\lim} \oplus c_0'$.)