Functional Analysis

Institute of Mathematics, LMU Munich – Spring Term 2012
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HOMEWORK ASSIGNMENT no. 6, issued on Tuesday 22 May 2012
Due: Wednesday 30 May 2012 by 6 pm in the designated "FA" box on the 1st floor
Info: `www.math.lmu.de/~michel/SS12_FA.html`

Each exercise is worth a full mark of 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in your solutions either in German or in English.

Exercise 21. (Completeness = nested balls shrink to a point. Examples of completions.)

(i) Let (X,d) be a metric space. Show that X is complete if and only if any sequence $\{K_{R_n}(x_n)\}_{n=1}^{\infty}$ of closed balls such that $K_{R_1}(x_1) \supset K_{R_2}(x_2) \supset K_{R_3}(x_3) \cdots$ and $R_n \xrightarrow{n\to\infty} 0$ has the property that $\bigcap_{n=1}^{\infty} K_{R_n}(x_n) = \{x\}$ for some $x \in X$.

(ii) Set $d(x,y) := |\arctan x - \arctan y|\ \forall x,y \in \mathbb{R}$. Show that $(\mathbb{R},d)$ is a non-complete metric space and find its completion.

(iii) Set $d(x,y) := |e^x - e^y|\ \forall x,y \in \mathbb{R}$. Show that $(\mathbb{R},d)$ is a non-complete metric space and find its completion.

(iv) On the set X of all segments $[a,b]$ of the real line ($a < b$) define $d([a,b],[c,d]) := |a-c| + |b-d|$. Show that (X,d) is a non-complete metric space and find its completion.

Exercise 22. (Norms are 1:1 with translation invariant, homogeneous metrics. Triangular inequality for norms $\Leftrightarrow$ closed unit ball is convex. Finite-dimensional norms are equivalent.)

(i) Let X be a vector space (on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). Show that there is a one-to-one correspondence between norms on X and metrics on X that are translation invariant and homogeneous. (Recall: a metric d is translation invariant when $d(x,y) = d(x+z,y+z)\ \forall x,y,z \in X$; d is homogeneous when $d(\alpha x, \alpha y) = |\alpha| d(x,y)\ \forall x,y \in X$ and for any scalar α.)

(ii) Let X be a vector space (on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) on which a function $p : X \to [0,\infty)$ is given with the two properties (a) $p(x) = 0 \Leftrightarrow x = 0$, (b) $p(\alpha x) = |\alpha| p(x)\ \forall x \in X,\ \forall \alpha \in \mathbb{K}$. Show that p is a norm if and only if $K := \{x \in X \,|\, p(x) \leqslant 1\}$ is convex.

(Recall: $E \subset X$ is convex when $tx + (1-t)y \in E$ for all $x,y \in E$ and $t \in [0,1]$.)

(iii) Let $N \in \mathbb{N}$, arbitrary. Show that there exists a constant $c_N > 0$ such that if $p : [0,1] \to \mathbb{R}$ is a polynomial of degree at most N then $p(\frac{1}{5}) \leqslant c_N \int_0^1 |p(x)| \mathrm{d}x$.

Exercise 23. (Banach is the same as absolutely convergent sequences converge. Application of the contraction principle to Banach spaces.)

(i) Let $(X, \| \|)$ be a normed space. Show that $(X, \| \|)$ is a Banach space if and only if every absolutely convergent series in X is convergent.

(Recall: the series $\sum_{n=1}^{\infty} x_n$ is convergent if $\left\{\sum_{n=1}^{N} x_n\right\}_{N=1}^{\infty}$ is a convergent sequence in X, whereas $\sum_{n=1}^{\infty} x_n$ is absolutely convergent if $\sum_{n=1}^{\infty} \|x_n\| < \infty$.)

(ii) The following are given: a Banach space $(X, \| \|)$, $m \in \mathbb{N}$, a continuous linear operator $T : X \to X$ with $\|T^m\| < 1$, $x_0 \in X$. Show that the equation

$$x - T(x) = x_0$$

has a unique solution $x \in X$.

(*Hint:* consider the m-th power of the map $x \to x_0 + T(x)$.)

Exercise 24. (Closure of ℓ^p in ℓ^q. Separability of ℓ^p.)

(i) Let $1 \leqslant p < q < \infty$. Show that ℓ^p is a proper dense subspace of ℓ^q.

(ii) Let $1 \leqslant p < \infty$. Find the closure of ℓ^p in ℓ^∞.

(iii) For which $p \in [1, \infty]$ is the Banach space ℓ^p separable? Justify your answer.