Functional Analysis

Institute of Mathematics, LMU Munich – Spring Term 2012
Prof. T. Ø. Sørensen Ph.D, A. Michelangeli Ph.D

HOMEWORK ASSIGNMENT no. 5, issued on Tuesday 15 May 2012
Due: Tuesday 22 May 2012 by 6 pm in the designated "FA" box on the 1st floor
Info: `www.math.lmu.de/~michel/SS12_FA.html`

Each exercise is worth a full mark of 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in your solutions either in German or in English.

Exercise 17. (Contractions on compact metric spaces. No surjective contractions on a compact. Iterated maps that are contractions. An inverse of Banach fixed point.)

(i) Let (X, d) be a non-empty compact metric space and let $\Phi : X \to X$ be such that
$$d(\Phi(x), \Phi(y)) < d(x, y), \qquad x \neq y.$$
Show that Φ has a unique fixed point x given precisely by $x = \lim_{n\to\infty} \Phi^n(x_0)$, x_0 arbitrary in X.

(ii) Find a compact metric space (X, d) and a function $\Phi : X \to X$ such that
$$d(\Phi(x), \Phi(y)) \leqslant d(x, y), \qquad x, y \in X,$$
while Φ has no fixed point.

(iii) Show that there is no contraction mapping from a compact metric space *onto* itself, i.e., surjective (assuming that the metric space has more than one point).

(iv) Let (X, d) be a complete metric space and let $\Phi : X \to X$ be such that Φ^m is a contraction for some integer $m \geqslant 1$. Show that Φ has a unique fixed point.

(v) Let (X, d) be a metric space such that any contraction $\Phi : E \to E$ on any non-empty closed subset E of X has a fixed point. Show that (X, d) is complete.

Exercise 18. (A non-compact unit ball. Examples of open, closure, interior in $C([0, 1])$.)

(i) Produce a sequence in the closed unit ball centred at zero of the metric space $(C([0, 1]), d_\infty)$ (see Problem 11(iii)) which does not admit any convergent subsequence.

(Note: this way you deduce that the closed unit ball in $(C([0, 1]), d_\infty)$ is not compact, which you know already by general arguments from Theorem 2.8 discussed in class.)

(ii) Show that the subspace $\mathcal{P}([0, 1])$ of all polynomials on $[0, 1]$ is not open in $(C([0, 1]), d_\infty)$.

(iii) Let $C^1([0, 1])$ denote the space of differentiable functions on $[0, 1]$ with continuous derivative. Find the interior of $C^1([0, 1])$ in $(C([0, 1]), d_\infty)$.

(iv) Let $C_0(\mathbb{R})$ denote the space of continuous functions whose support is compact in $\mathbb{R}$. (The SUPPORT of a continuous function on $\mathbb{R}$ is the set of points where the function does not vanish.) Consider the metric space $(C_b(\mathbb{R}), d_\infty)$ with
$$C_b(\mathbb{R}) \;:=\; \big\{ f : \mathbb{R} \to \mathbb{R} \,|\, f \text{ is continuous and } |f(x)| \leqslant C_f \text{ for some } C_f \geqslant 0 \big\}$$
and $d_\infty(f, g) := \sup_{x\in\mathbb{R}} |f(x) - g(x)|\ \forall f, g \in C_b(\mathbb{R})$. Find the closure of $C_0(\mathbb{R})$ in $(C_b(\mathbb{R}), d_\infty)$.

Exercise 19. (Topologies with the same convergent sequences. Equivalent metrics. Equivalent norms.)

(i) If the topological spaces $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$ (that as a set are the same) have the same convergent sequences, are the two topologies the same? Justify your answer.

(Note: compare this with Problem 12(i).)

(ii) Consider the metric spaces $(\mathbb{R}, d_1)$ and $(\mathbb{R}, d_2)$, where $d_1(x, y) := |x - y|$ and $d_2(x, y) := |\phi(x) - \phi(y)|\ \forall x, y \in \mathbb{R}$, where $\phi(x) := \frac{x}{1+|x|}$. Show that d_1 and d_2 induce the same topology on $\mathbb{R}$ (i.e., they are topologically equivalent) but $(\mathbb{R}, d_1)$ is complete whereas $(\mathbb{R}, d_2)$ is not.

(iii) Let $(X, \| \|_1)$ and $(X, \| \|_2)$ be two normed spaces. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the topologies induced on X by the norms $\| \|_1$ and $\| \|_2$ respectively. Show that $\mathcal{T}_2$ is finer than $\mathcal{T}_1$ (i.e., $\mathcal{T}_1 \subset \mathcal{T}_2$) if and only if $\exists\, C > 0$ such that $\|x\|_1 \leqslant C\|x\|_2$ for all $x \in X$.

(iv) Show that two norms $\| \|_1$ and $\| \|_2$ on a space X induce the same topology on X (i.e., they are topologically equivalent) if and only if $c\|x\|_2 \leqslant \|x\|_1 \leqslant C\|x\|_2$ for all $x \in X$ and for some constants $c, C > 0$.

(Note: compare this with the notion of topologically equivalent *metrics*, Problem 12.)

Exercise 20. (Examples of open/closed in ℓ^p.)

(i) Let $a = \{a_n\}_{n=1}^{\infty}$ be a given sequence in $(0, +\infty)$ and set

$$S^{(a)} := \Big\{\, x = (x_1, x_2, x_3, \dots) \text{ such that } \sum_{n=1}^{\infty} |x_n|^2 < \infty \text{ and } |x_n| < a_n\ \forall n \,\Big\} \subset \ell^2.$$

Find a necessary and sufficient condition on $\inf_n a_n$ in order $S^{(a)}$ to be open in ℓ^2.

(ii) Let $p \in [1, \infty)$ and set

$$E := \Big\{\, x = (x_1, x_2, x_3, \dots) \in \ell^p \text{ such that } \sum_{n=1}^{\infty} x_n = 0 \,\Big\} \subset \ell^p.$$

For which p is E closed in ℓ^p? Justify your answer.

(iii) Is the space

$$c_0 := \Big\{\, x = (x_1, x_2, x_3, \dots) \mid x_n \in \mathbb{C} \text{ and } \lim_{n\to\infty} x_n = 0 \,\Big\}$$

closed in ℓ^∞? Justify your answer.