Functional Analysis

Institute of Mathematics, LMU Munich – Spring Term 2012
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HOMEWORK ASSIGNMENT no. 4, issued on Tuesday 8 May 2012
Due: Tuesday 15 May 2012 by 6 pm in the designated "FA" box on the 1st floor
Info: `www.math.lmu.de/~michel/SS12_FA.html`

Each exercise is worth a full mark of 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in your solutions either in German or in English.

Exercise 13. (Open covers without finite subcover. Closed and bounded is less then compact, if the space is not complete. Hausdorff non-compact, compact non-Hausdorff.)

(i) Regard the interval $(0, 1)$ as a subspace of the topological space $\mathbb{R}$ with respect to the usual Euclidean metric topology. Produce an open cover of $(0, 1)$ which does not admit finite subcover.

(Note: of course you *already* knew that $(0, 1)$ is not compact in $\mathbb{R}$ because it is not closed.)

(ii) Consider, in the metric space $\mathbb{Q}$ of the rationals equipped with the Euclidean metric, the subspace $E := [0, 1] \cap \mathbb{Q}$. Show that E is closed and bounded in $\mathbb{Q}$, and produce an open cover of E which does not admit finite subcover.

(iii) Produce an example of a Hausdorff non-compact space, a finite compact non-Hausdorff space, an infinite compact non-Hausdorff space.

Exercise 14. (Continuous bijections are not homeomorphisms in general. They are between compact Hausdorff spaces.)

(i) Show that if $\phi : X \to Y$ is a continuous function between topological spaces and if X is compact then $f(X)$ is a compact subset of Y.

(ii) Let $X := [0, 1)$ and $Y := \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\}$, equipped with the relative topology induced by $\mathbb{R}$ and $\mathbb{R}^2$ respectively. Show that the map $\phi : X \to Y$, $\theta \mapsto (\cos 2\pi\theta, \sin 2\pi\theta)$ is a continuous bijection but not a homeomorphism. Are X and Y homeomorphic?

(iii) Show that if $\phi : X \to Y$ is a continuous bijection between a compact space X and a Hausdorff space Y then ϕ is a homeomorphism.

(iv) Follow-up to (iii): if X is compact and Hausdorff and Y is compact, is the continuous bijection $\phi : X \to Y$ necessarily a homeomorphism? Justify your answer.

Exercise 15. (Homeomorphisms do not preserve completeness. A metric making the irrationals complete and homeomorphic to the Euclidean non-complete irrationals.)

(i) Produce an example of two metric spaces that are homeomorphic as topological spaces but one is complete whereas the other is not.

(Note: you are supposed to provide an *easy* example: some two uncountably infinite spaces that have been in front of you since ever. Also, the example discussed in (ii) and (iii) below will not be accepted as an answer here.)

(ii) Equip $\mathbb{R}\backslash\mathbb{Q}$ with the metric

$$d(x,y) := |x-y| + \sum_{n=1}^{\infty} \frac{1}{2^n} \min\Big\{\, 1\,,\, \Big| \frac{1}{\min\limits_{j\leqslant n} |x-q_j|} - \frac{1}{\min\limits_{j\leqslant n} |y-q_j|} \Big| \,\Big\}\,.$$

where $\{q_j\}_{j=1}^{\infty}$ is an enumeration of the rationals. (The fact that d is indeed a metric was checked in the solution to Exercise 12(ii). Note that since one is considering here the irrationals only, there is no need to assume any convention $1/0=\infty$ and the like.) Show that $(\mathbb{R}\backslash\mathbb{Q}, d_{\mathrm{Eucl}})$, the metric space of the irrationals equipped with the usual Euclidean metric, and $(\mathbb{R}\backslash\mathbb{Q}, d)$ are homeomorphic.

(iii) Show that $(\mathbb{R}\backslash\mathbb{Q}, d_{\mathrm{Eucl}})$ is not complete whereas $(\mathbb{R}\backslash\mathbb{Q}, d)$ is.

Exercise 16. ($\mathbb{R}$ is isometrically homeomorphic to the punctured disk, its completion is the disk.)

Let E be the subspace of $\mathbb{R}^2$ consisting of the circle centred at $(0,\frac{1}{2})$ of radius $\frac{1}{2}$ without the point $(0,1)$, equipped with the relative topology induced by $\mathbb{R}^2$. Define $h:\mathbb{R}\to E$ so that $h(x)$ is the intersection of E with the line segment from $(x,0)$ to $(0,1)$.

(i) Show that h is a homeomorphism from $\mathbb{R}$ to E.

(ii) Show that $d(x,y) := |h(x)-h(y)|$, $x,y\in\mathbb{R}$, define a metric d on $\mathbb{R}$ which is equivalent to the usual Euclidean metric.

(iii) Show that $(\mathbb{R},d)$ is not complete and identify its completion.

(iv) Follow-up to (iii): show that the completion of $(\mathbb{R},d)$ is compact.