Institute of Mathematics, LMU Munich – Spring Term 2012
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HOMEWORK ASSIGNMENT no. 2, issued on Tuesday 24 April 2012
Due: Wednesday 2 May by 12 pm in the designated "FA" box on the 1st floor
Info: `www.math.lmu.de/~michel/SS12_FA.html`

Each exercise is worth a full mark of 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in your solutions either in German or in English.

Exercise 5. (Elementary facts on closure, interior, boundary with respect to complement, inclusion, union, intersection.)

Let X be a topological space and $E \subset X$ and let $E, F \subset X$.

(i) Show that $\partial E = \partial(X \setminus E)$.

(ii) Show that $E \subset F \Rightarrow \mathring{E} \subset \mathring{F}$ and $\overline{E} \subset \overline{F}$.

(iii) Show that $\big(\mathring{E}\big)^{\circ} = \mathring{E}$.

(iv) Show that $\overline{E \cup F} = \overline{E} \cup \overline{F}$.

(v) Show that $\overline{E \cap F} \subset \overline{E} \cap \overline{F}$ and give an example of strict inclusion (thus disproving the opposite inclusion).

(vi) Show that $(E \cup F)^{\circ} \supset \mathring{E} \cup \mathring{F}$ and give an example of strict inclusion.

(vii) Show that $(E \cap F)^{\circ} = \mathring{E} \cap \mathring{F}$.

(viii) Show that $\partial(E \cup F) \subset \partial E \cup \partial F$ and give an example of strict inclusion.

(ix) Give examples of $\partial(E \cap F) \subsetneq \partial E \cap \partial F$ and $\partial(E \cap F) \supsetneq \partial E \cap \partial F$.

Exercise 6. (Base for a topology. Characterisation and relation with a basis of neighbourhoods.)

Given a topological space $(X, \mathcal{T})$, consider a family $\mathscr{B} \subset \mathcal{T}$ of opens such that any $\mathcal{O} \in \mathcal{T}$ is the union of opens in $\mathscr{B}$. Clearly, such a family always exists and is called BASE for the topology $\mathcal{T}$.

(i) Show that a family $\mathscr{B}$ of open subsets of a topological space $(X, \mathcal{T})$ is a base for $\mathcal{T}$ *if and only if* for each $x \in X$ the collection $\mathscr{B}_x := \{B \in \mathscr{B} \,|\, B \ni x\}$ is a basis of neighbourhoods at x, i.e. (recall the definition in class), for every neighbourhood $\mathcal{N} \ni x$ there exists $B \in \mathscr{B}_x$ such that $x \in B \subset \mathcal{N}$.

(ii) Show that a family $\mathscr{B}$ of subsets of a set X is a base for a topology of X *if and only if* $\mathscr{B}$ has the following two properties:

(a) each $x \in X$ belongs to some $B \in \mathscr{B}$,

(b) if $x \in B_1 \cap B_2$ for $B_1, B_2 \in \mathscr{B}$, then there is a $B_3 \in \mathscr{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

(iii) Let $\mathscr{B}$ be the family of subsets of $\mathbb{R}$ of the form $[a, b)$, $a, b \in \mathbb{R}$, $a < b$, together with $\emptyset$. Show that $\mathscr{B}$ is a base for a topology in $\mathbb{R}$.

Exercise 7. (Characterisation of closure and closed sets.)

Let X be a topological space and $E \subset X$.

(i) Show that $\overline{E} = E \cup \{\text{limit points of } E\}$

(ii) Show that $\overline{E} = X \setminus (\overset{\circ}{X \setminus E})$ (i.e., the complement of the exterior of E).

(iii) Show that the following properties are equivalent:

  (a) E is closed

  (b) $\overline{E} = E$

  (c) $\partial E \subset E$

  (d) $E = \{\text{adherent points to } E\}$

  (e) For each $x \in X$, if every neighbourhood of x intersects E, then $x \in E$

  (f) $E \supset \{\text{limits points of } E\}$.

Assume now in addition that the topology on X is such that sets consisting of one point are closed.

(iv) Show that $\overline{E} = \{\text{limit points of } E\} \sqcup \{\text{isolated points of } E\}$

where $\sqcup$ is the disjoint union and $x \in E$ is called ISOLATED POINT if there exists a neighbourhood $\mathcal{N} \ni x$ such that $E \cap \mathcal{N} = \{x\}$.

(v) Show that the set of limit points of E is closed.

Exercise 8. (Limits of sequences do not identify the closure, in general. The case of the co-countable topology.)

(i) Let X be a topological space, $E \subset X$, $\{x_n\}_{n=1}^{\infty}$ be a sequence in E convergent to some $x \in X$. Show that $x \in \overline{E}$.

(ii) Let X be a set and let $\mathcal{T}$ be the family of subsets $\mathcal{O}$ of X such that $X \setminus \mathcal{O}$ is at most countable, together with the empty set $\emptyset$. Show that $\mathcal{T}$ is a topology.

(iii) Find *all* convergent sequences in the topological space $(X, \mathcal{T})$ introduced in (ii).

(iv) Equip $\mathbb{R}$ (or any uncountable set X) with the topology $\mathcal{T}$ introduced in (ii). Produce a subset $E \subset \mathbb{R}$ such that $\overline{E}$ contains (among others) points that are *not* limits of sequences in E.