INDIVIDUAL PROJECT NO.2, issued on Tuesday 21 June 2011
Due: Wednesday 6 July 2011 in the exercise session
Info: `www.math.lmu.de/~michel/SS11_MQM2.html`

Work out individually the details of the problem outlined in the scheme below. Results and techniques discussed in the class as well as in the tutorial and exercise sessions will be needed.

Part I. Consider the operator

$$A = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}\,, \qquad \mathcal{D}(A) = \{f \in C^{2011}([0,1]) \,|\, f'(0) = 0 = f'(1)\}$$

on $L^2[0,1]$.

(i) Show that A is symmetric and non-negative.

(ii) Determine the form domain $\mathcal{D}(q_A)$ of A and the action of the closed, non-negative quadratic form q_A associated with A. ($\mathcal{D}(q_A)$ should be identified by a "known" space, you cannot just repeat the definition.)

(iii) Determine the Friedrichs' extension A^F of A, i.e., its domain and its action.

(iv) Determine A^* (domain and action).

(v) Show that A is essentially self-adjoint and that $\overline{A} = A^* = A^F$.

Part II. Consider the operator

$$A = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}\,, \qquad \mathcal{D}(A) = \{f \in H^2(\mathbb{R}) \,|\, f(0) = 0\}$$

on $L^2(\mathbb{R})$.

(i) Show that A is symmetric, closed, and non-negative.

(ii) Show that

$$A^* = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}\,, \qquad \mathcal{D}(A^*) = \{f \in H^2(\mathbb{R}\setminus\{0\}) \,|\, f(0^+) = f(0^-)\}\,.$$

The notation is $\varphi(0^+) := \lim_{x\downarrow 0}\varphi(x)$, $\varphi(0^-) := \lim_{x\uparrow 0}\varphi(x)$.

(iii) Show that A has deficiency indices $n_+(A) = n_-(A) = 1$ and determine the normalised functions ψ_+, ψ_- spanning the deficiency spaces $\mathcal{H}_+$ and $\mathcal{H}_-$ respectively.

(iv) From (iii), the theory of self-adjoint extensions tells you that all self-adjoint extensions of A have the form A_θ, $\theta \in [0, 2\pi)$, where

$$\begin{aligned} \mathcal{D}(A_\theta) &= \{f + c\psi_+ + ce^{\mathrm{i}\theta}\psi_- \mid f \in \mathcal{D}(A),\, c \in \mathbb{C}\} \\ A_\theta(f + c\psi_+ + ce^{\mathrm{i}\theta}\psi_-) &= -f'' + \mathrm{i}\, c\, \psi_+ - \mathrm{i}\, c\, e^{\mathrm{i}\theta}\psi_- \,. \end{aligned}$$

Show that if $g \in \mathcal{D}(A_\theta)$ then

$$g'(0^+) - g'(0^-) = \alpha g(0) \qquad (*)$$

where

$$\alpha := -2\cos\Big(\frac{\theta}{2}\Big) \Big/ \cos\Big(\frac{\theta}{2} - \frac{\pi}{4}\Big) . \qquad (**)$$

(v) Re-label the extension A_θ of A described in (iv) as A_α with α defined in $(**)$. Show that if $\alpha < 0$ then A_α has precisely one negative, simple eigenvalue $\lambda = -\alpha^2/4$ with (non-normalised) eigenfunction $e^{\alpha|x|/2}$, whereas if $\alpha > 0$ then A_α has no eigenvalues.

(vi) Fix $\alpha \in \mathbb{R}$ and let $\psi \in \mathcal{D}(A_\alpha)$. Show that

$$A_\alpha \psi = -\psi'' + \alpha\, \psi(0)\, \delta$$

holds in distributional sense. (Here δ is the delta distribution.)

Remark: What you found in (vi) is suggestive of the fact that in some sense

$$A_\alpha = \text{“}-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \alpha\, \delta(x)\text{”} .$$

In the following two items this identification is discussed further by considering the quadratic form associated with “$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \alpha\,\delta(x)$” and the corresponding Friedrichs' self-adjoint operator.

(vii) For any $\alpha \in \mathbb{R}$ consider the quadratic form q_α on $L^2(\mathbb{R})$ defined by

$$q_\alpha(f, g) = \int_{\mathbb{R}} \overline{f'(x)}\, g'(x)\, \mathrm{d}x \;+\; \alpha\, \overline{f(0)}\, g(0) , \qquad \mathcal{D}(q_\alpha) = H^1(\mathbb{R}) .$$

Show that q_α is bounded below and closed.

(viii) (★) *OPTIONAL. This part is for extra credit and is meant only for dedicated students.* Owing to (vii) and to the Friedrichs' extension theorem, q_α gives rise to a unique self-adjoint operator H_α with domain $\mathcal{D}(H_\alpha)$ contained in $\mathcal{D}(q_\alpha)$ and such that H_α is bounded below by the same bound as q_α. Prove that actually $H_\alpha = A_\alpha$ by showing that

$$\mathcal{D}(A_\alpha) = \Big\{ \Big(1 + \frac{\alpha}{2}\,|x|\Big)\chi + \varphi \,\Big|\, \chi \in C_0^\infty(\mathbb{R})\,, \chi'(0) = 0\,, \varphi \in H^2(\mathbb{R})\,, \varphi(0) = 0 \Big\} = \mathcal{D}(H_\alpha) .$$

(*Hint:* use the condition $(*)$ to identify $\mathcal{D}(A_\alpha)$ and use the construction of $\mathcal{D}(H_\alpha)$ discussed in class and in the tutorials.)