Advanced Mathematical Quantum Mechanics

TMP Programme, Munich – Summer Term 2011

INDIVIDUAL PROJECT NO.1, issued on Thursday 26 May 2011
Due: Wednesday 15 June 2011 in the exercise session
Info: `http://www.mathematik.uni-muenchen.de/~michel/SS11_MQM2.html`

Work out individually the details of the problem outlined in the scheme below. Results and techniques discussed in the class as well as in the tutorial and exercise sessions will be needed.

Part I. Consider the operator $A : L^2[0,1] \to L^2[0,1]$ defined by

$$(Af)(x) \;=\; \int_0^1 K(x,y)f(y)\,\mathrm{d}y \qquad \text{a.e. in } x \in [0,1]$$

where

$$K(x,y) \;:=\; \begin{cases} x(y-1) & \text{if } \; 0 \leqslant x \leqslant y \leqslant 1 \\ y(x-1) & \text{if } \; 0 \leqslant y \leqslant x \leqslant 1\,. \end{cases}$$

(i) Show that A is a Hilbert-Schmidt, self-adjoint operator and compute its Hilbert-Schmidt norm $\|A\|_{\mathrm{HS}}$.

(ii) Determine $\sigma(A)$, eigenvalues, eigenfunctions, and the spectral decomposition of A.

(iii) Compute the operator norm of A, $\|A\|$.

(iv) Let $\lambda \in \rho(A)$. Compute $(\lambda - A)^{-1}g$ where g is the $L^2[0,1]$-function defined by $g(x) = x$.

(v) *Facultative.* Show that $A \leqslant 0$ and that A is a trace-class operator, and compute its trace-class norm $\|A\|_{\mathrm{TC}}$.

(Notation: $\rho(A)$ = resolvent set of A, $\sigma(A)$ = spectrum of A.)

Part II. Determine all solutions $f \in L^2[0,1]$ and $\lambda \in \mathbb{C}$ to the equation

$$f(x) - \lambda \int_0^1 K(x,y)f(y)\,\mathrm{d}y \;=\; \cos \pi x \qquad \text{a.e. in } x \in [0,1]$$

where

$$K(x,y) \;:=\; \begin{cases} y(x+1) & \text{if } \; 0 \leqslant x \leqslant y \leqslant 1 \\ x(y+1) & \text{if } \; 0 \leqslant y \leqslant x \leqslant 1\,. \end{cases}$$

Part III. Consider the operator $A_0 : \mathcal{D}(A_0) \to L^2[0,1]$ defined on the domain

$$\mathcal{D}(A_0) \;=\; \{f \in C^1([0,1]) \,|\, f(0) = f(1) = 0\}$$

by

$$A_0 f \;=\; -\mathrm{i}f' + xf\,, \qquad\qquad f \in \mathcal{D}(A_0)\,.$$

(i) Show that A_0 is densely defined, symmetric, and unbounded on $L^2[0,1]$.

(ii) Determine the operator A_0^* (the adjoint of A_0), i.e., determine its domain and its action.

(iii) Determine the operator $\overline{A_0}$ (the closure of A_0), i.e., determine its domain and its action.

(iv) Prove that A_0 is neither closed nor self-adjoint.

(v) Prove that A_0 has no eigenvalues.

(vi) Compute the deficiency indices n_+ and n_- and determine the deficiency spaces $\mathcal{D}_+$ and $\mathcal{D}_-$ of A_0. Recall: $n_\pm = \dim(\mathcal{D}_\pm)$, $\mathcal{D}_\pm = \mathrm{Ker}(A_0^* \mp \mathrm{i})$.

(vii) Determine all self-adjoint extensions of A_0 (domain and action).

(viii) Show that the operator $A : \mathcal{D}(A) \to L^2[0,1]$ defined on the domain

$$\mathcal{D}(A) \;=\; \{f \in H^1[0,1] \,|\, f(0) = f(1)\}$$

by

$$Af \;=\; -\mathrm{i}f' + xf\,, \qquad\qquad f \in \mathcal{D}(A)\,.$$

is a self-adjoint extension of A_0 and determine its eigenvalues and eigenfunctions.