Functional Analysis – Problems in the class, sheet 10

Mathematisches Institut der LMU – SS2010
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The problems in the class are discussed interactively in the weekly exercise/tutorial sessions. You are not required to hand in their solution. You are encouraged to think them over and to solve them. Being able to solve them is essential for your preparation towards the final exam. Further info at `www.math.lmu.de/~michel/SS10_FA.html`.

Problem 37. Let X be a vector space equipped with two distinct norms $\| \; \|_1$ and $\| \; \|_2$ so that both $(X, \| \; \|_1)$ and $(X, \| \; \|_2)$ are Banach spaces. Show that if $\exists\, C$ such that $\|x\|_1 \leqslant C\|x\|_2$ $\forall x \in X$ then $\exists\, D$ such that $\|x\|_2 \leqslant D\|x\|_1$ $\forall x \in X$ and therefore the two norms are equivalent. (*Hint:* use the inverse mapping theorem.)

Problem 38. Let X be a Banach space. The dual space of the dual of X, i.e., the dual space of X^*, is called the BIDUAL of X and is denoted by X^{**}.

(i) For each $x \in X$ consider the map $E_x : X^* \to \mathbb{C}$ such that $E_x(\phi) := \phi(x)$ for all $\phi \in X^*$. Prove that $E_x \in X^{**}$.

(ii) Prove that $\|E_x\|_{X^{**}} = \|x\|_X$ and therefore $x \mapsto E_x$ is an isometry of X *into* X^{**}. (*Hint:* use Hahn-Banach.)

The isometry $x \mapsto E_x$ defined above is called the CANONICAL EMBEDDING of X into X^{**}. If the canonical embedding $X \to X^{**}$ is surjective, then the space X is said to be REFLEXIVE. (Note: reflexivity is not just $X \cong X^{**}$, but $X \xrightarrow[E]{\cong} X^{**}$ via the canonical embedding!)

(iii) Decide which of these Banach spaces, with their natural norm, are reflexive:

$$L^p(\mathbb{R}^n) \quad (1 < p < \infty)\,, \qquad L^1(\mathbb{R}^n)\,, \qquad \text{a Hilbert space } \mathcal{H}\,, \qquad c_0\,.$$

Problem 39. Prove that there exists no normed space X such that its dual X^* is isomorphic as a Banach space to c_0, where c_0 is the space of sequences going to 0 and equipped with the norm $\| \; \|_\infty$. (*Hint:* consider the canonical embedding of X into X^{**}.)

Problem 40. Prove that the space c_{00} of the sequences of complex numbers having finitely many non-zero elements cannot be equipped with a norm that would make it a Banach space. (*Hint:* use Baire's category theorem.)