Functional Analysis – Exercise sheet 10

Mathematisches Institut der LMU – SS2010
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Due: Monday 12 July 2010 by 12 p.m. in the designated "Funktionalanalysis" box on the 1st floor
Students who will be attending the Mon 12 July tutorial have to hand in their solution sheets at 10:15 in class.
Info: `www.math.lmu.de/~michel/SS10_FA.html`

The full mark in each exercise is 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in the solutions either in German or in English.

Exercise 37. Prove that every weakly convergent sequence in ℓ^1 converges in norm. (*Hint:* In the case when $x_n \to 0$ weakly, by passing to a subsequence x_{n_r} one can find disjoint finite intervals I_r of the integers such that "most of the norm" of x_{n_r} is concentrated on I_r.)

Exercise 38. Let $\mathcal{H}$ be a separable Hilbert space.

(i) Prove that the weak topology in the unit ball $\mathcal{B}_1 = \{x \in \mathcal{H} \,|\, \|x\| \leqslant 1\}$ is metrizable, i.e., there exists a metric d in $\mathcal{B}_1$ such that a set is open in the sense of the weak topology if and only if it is open in the sense of the metric d.

(ii) Prove that the weak topology in the whole $\mathcal{H}$ is not metrizable if $\dim \mathcal{H} = \infty$. (*Hint:* prove that any non-empty, weakly open set is unbounded in $\mathcal{H}$, then show that there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathcal{H}$ that is weakly convergent to zero but whose norm blows up.)

Exercise 39.

(i) Let V a proper subspace of a normed space X. Show that V has empty interior.

(ii) Let $\mathcal{P}$ be the vector space of all real polynomials in one variable. Equip $\mathcal{P}$ with an arbitrary norm $\|\ \|$. Show that $(\mathcal{P}, \|\ \|)$ is *not* a Banach space. (*Hint:* use Baire's category theorem.)

Exercise 40. Let $\mathcal{H}$ be a separable Hilbert space and let $\{\varphi_n\}_{n=1}^{\infty}$ be an orthonormal basis. Let $\{\psi_n\}_{n=1}^{\infty}$ be a collection of elements in $\mathcal{H}$. Prove that the following two statements are equivalent:

(a) $\psi_n \xrightarrow{n\to\infty} 0$ weakly

(b) $\langle \varphi_m, \psi_n \rangle \xrightarrow{n\to\infty} 0$ for each $m = 1, 2, 3, \dots$ and $\|\psi_n\| < C$ for all n and some constant C independent of n.