Functional Analysis II – Problem sheet 8

Mathematisches Institut der LMU – SS2009
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Handout: 16.06.2009
Due: Tuesday 23.06.2009 by 1 p.m. in the "Funktionalanalysis II" box
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Grader: Ms. S. Sonner – Übungen on Wednesdays, 4,30 - 6 p.m., room C-111

Exercise 21. Let $x_0 \in \mathbb{R}^d$, $x_0 \neq 0$, $d \geqslant 1$. Let $T_{x_0} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be the operator of translation by x_0, that is, $(T_{x_0}\psi)(x) := \psi(x - x_0)$ $\forall \psi \in L^2(\mathbb{R}^d)$. Prove that

(a) T_{x_0} is a unitary operator on $L^2(\mathbb{R}^d)$,

(b) $\mathrm{Spec}(T_{x_0}) = \{\lambda \in \mathbb{C} \,:\, |\lambda| = 1\}$,

(c) $\mathrm{Spec}_{\mathrm{pp}}(T_{x_0}) = \emptyset$.

(*Hint:* for one of the two inclusions in part (b) you may refer to Example 1.54 given in the class, for the other inclusion you may apply the Weyl's criterion.)

Exercise 22. Let $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$, the Schwartz space of smooth functions with rapid decrease in dimension $d \geqslant 1$.

(a) Prove that $\varphi * \psi \in \mathcal{S}(\mathbb{R}^d)$, i.e., the convolution of two Schwartz functions is a Schwartz function.

(b) Prove that the Fourier transform[1] $\mathcal{F}$ of the convolution is given by

$$\mathcal{F}(\varphi * \psi) \;=\; (2\pi)^{d/2}\mathcal{F}(\varphi)\mathcal{F}(\psi)$$

Exercise 23. Let $\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ be the Fourier transform operator on the Schwartz space of smooth functions with rapid decrease in dimension $d \geqslant 1$. Let $\varphi(x) := e^{-x^2/2}$. Prove that $\mathcal{F}\varphi = \varphi$.

[1] remember that the convention adopted in the class is $(\mathcal{F}f)(\xi) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} f(x)e^{-ix\xi}\mathrm{d}x$