Advanced Mathematical Quantum Mechanics

Mathematisches Institut der LMU – SS2009
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Individual project no. 2: Relativistic Lieb-Thirring inequality and relativistic kinetic energy inequality

- ✓ Work out individually the details of the problem outlined in the scheme below.
- ✓ Results and techniques discussed in the class as well as in the tutorial sessions and in the weekly homeworks will be needed.
- ✓ Questions, info, further clarifications: Alessandro, usual times and places.
- ✓ Please return your completed project by June, Friday 26.

Part 1. [*Relativistic Lieb-Thirring inequality*] The goal of this problem is the proof of the following statement (d is a positive integer and the potential $V : \mathbb{R}^d \to \mathbb{R}$ is assumed to satisfy the standard conditions discussed in the class for selfadjointness, boundedness below and existence of the ground state of the corresponding pseudo-relativistic one-body Hamiltonian[1]).

> *Fix $\gamma > 0$ and assume that the negative part of the potential, V_-, satisfies the condition $V_- \in L^{\gamma+d}(\mathbb{R}^d)$. Assume that $A \in L^2_{loc}(\mathbb{R}^d;\mathbb{R}^d)$. Let $E_0 \leqslant E_1 \leqslant E_2 \leqslant \cdots$ be the non-positive eigenvalues, if any, of $|-i\nabla + A(x)| + V$ in $\mathbb{R}^d$. Then there is a finite constant $L_{\gamma,d}$, independent of V, such that*

$$\sum_{j\geqslant 0} |E_j|^\gamma \;\leqslant\; L_{\gamma,d} \int_{\mathbb{R}^d} V_-^{\gamma+d}(x)\,\mathrm{d}x\,. \tag{LT-rel}$$

You can use the following ingredients without proof.

- The *Birman-Schwinger principle.* As it is stated in Theorem 4.1, Section 5, of the handout `www.math.lmu.de/~lerdos/WS08/QM/lt.pdf`, it holds in the relativistic case too, i.e., if $h = -\Delta + V$ is changed to $h = |-i\nabla + A| + V$ and $K_e := \sqrt{V_-}\,\big(|-i\nabla + A| + e\big)^{-1}\sqrt{V_-}$, $e > 0$.
- Lemma 5.1 in the same handout.

[1] $V \in L^d(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ if $d \geqslant 2$ and $V \in L^{1+\varepsilon}(\mathbb{R}) + L^\infty(\mathbb{R})$ if $d = 1$, together with the condition of vanishing at infinity, i.e., $|\{x : |V(x)| > a\}| < \infty \;\forall a > 0$

- The diamagnetic inequality

$$\big|\big(|-i\nabla+A|+e\big)^{-1}(x,y)\big| \;\leqslant\; \big(|\nabla|+e\big)^{-1}(x,y) \qquad\qquad x,y\in\mathbb{R}^d$$

  (this is the relativistic counterpart of Exercise 44 in the QM class last semester).

Remark. Inequality (LT-rel) holds for $\gamma=0$ when $d\geqslant 2$ (borderline case) but the proof would be different.

Part 2. [*Relativistic kinetic energy inequality*] As a corollary of (LT-rel), prove the following.

> *Fix $d=3$. Assume that $A\in L^2_{loc}(\mathbb{R}^d;\mathbb{R}^d)$. For any $\Psi\in L^2(\mathbb{R}^{3N})$ denote by ϱ_ψ the one-body density $\varrho_\psi(x):=N\int_{\mathbb{R}^{3(N-1)}}|\Psi(x,x_2,\dots,x_N|^2\mathrm{d}x_2\cdots\mathrm{d}x_N$. Then there exists a universal constant K such that*
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> $$\Big(\Psi,\sum_{j=1}^N\big|-i\nabla_{x_j}+A(x_j)\big|\,\Psi\Big)\;\geqslant\;K\int_{\mathbb{R}^3}\varrho_\psi(x)^{4/3}\mathrm{d}x \qquad (\blacklozenge)$$
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> *for any $\Psi\in L^2_{\mathrm{asym}}(\mathbb{R}^{3N})=\bigwedge_1^N L^2(\mathbb{R}^3)$.*

Inequality ($\blacklozenge$) is the analogue of the non-relativistic kinetic energy inequality discussed in Section 3 of the handout `www.math.lmu.de/~lerdos/WS08/QM/lt.pdf`. It is understood that both sides of (LT-rel) can be also infinite.