Mathematical Statistical Physics, 2015
Homework Problems, LMU

Issued: June 10, 2015; deadline for handing in the solutions:
June 17, 2015, 10 pm (22:00)

23. Let $\mathcal{H}$ stand for the one-particle Hilbert space and H for the one-particle Hamiltonian. Define a $\star$ automorphism τ_t on $\mathcal{A}_{\mathrm{CAR}}(\mathcal{H})$ by $\tau_t(a^\sharp(f)) := a^\sharp(e^{\mathrm{i}Ht}f)$ for $t \in \mathbb{R}$. Show that for a (β, τ_t) KMS state ω the following relation holds

$$\omega(a^\star(f_1)\cdots a^\star(f_n)a(g_1)\cdots a(g_m)) = \delta_{n,m}\det\big(\big(\langle g_i, \frac{\exp(-\beta H)}{1+\exp(-\beta H)}f_j\rangle\big)\big) \tag{58}$$

and conclude that there exists at most one (β, τ_t) KMS state.

24. Let $\mathcal{A}_{\mathrm{SD}}(\mathcal{H})$ be the completion of the $\star$ algebra generated by the operators $b(f), b^\star(f)$ and a unit, where $f \in \mathcal{H}$ and where $\mathcal{H}$ is a Hilbert space. Here, $f \mapsto b(f)$ is assumed to be linear, and $b(f)b^\star(g) + b^\star(g)b(f) = \langle g,\, f\rangle$. In addition, $b^\star(f) = b(\Gamma f)$ holds, where Γ is an antiunitary involution on $\mathcal{H}$, viz., $\Gamma^2 = 1$ and $\langle \Gamma f, \Gamma g\rangle = \langle g, f\rangle$ for $f, g \in \mathcal{H}$. (As an example for Γ, one may take the complex conjugation that in field theory exchanges the particles (i.e., positive energy solutions) with the antiparticles (i.e., negative energy solutions). We call a projection $P \in \mathcal{B}(\mathcal{H})$ a "basis projection" if $\Gamma P \Gamma = 1 - P$, and a state ω on $\mathcal{A}_{\mathrm{SD}}$ a "quasi-free state" if

$$\omega(b(f_1)\cdots b(f_{2n+1})) = 0, \tag{59}$$

and

$$\omega(b(f_1)\cdots b(f_{2n})) = (-1)^{n(n-1)/2}\sum_{\pi\in S_<}\mathrm{sign}(\pi)\Pi_{j=1}^n\omega(b(f_{\pi(j)})b(f_{\pi(j+n)})) \tag{60}$$

for $n \in \mathbb{N}$. Here $S_<$ denotes the set of permutations π of $2n$ elements that obey $\pi(j) < \pi(j+1)$ for $j \in \{1, \ldots, n-1\}$, while $\pi(j) < \pi(j+n)$ holds for $j \in \{1, \ldots, n\}$.

(i) Construct a $\star$ isomorphism $\mathcal{A}_{\mathrm{SD}}(\mathcal{H}) \to \mathcal{A}_{\mathrm{CAR}}(P\mathcal{H})$ for some basis projection P.

(ii) Show that for every state ω over $\mathcal{A}_{\mathrm{SD}}(\mathcal{H})$ there exists an operator $S \in \mathcal{B}\mathcal{H}$ such that

$$\omega(b(f)^\star\, b(g)) = \langle f, Sg \rangle \tag{61}$$

and S satisfies

$$0 \leq S = S^\star \leq 1, \quad S + \Gamma S \Gamma = 1. \tag{62}$$

25. For the system discussed in problem 24, prove that for each S that obeys (62) there exists a unique quasi-free state ω for which (61) holds.
Hint: Consider the operator P_S on $\mathcal{H} \oplus \mathcal{H}$

$$\rho_\alpha = \begin{pmatrix} S & S^{1/2}(1-S)^{1/2} \\ S^{1/2}(1-S)^{1/2} & 1-S \end{pmatrix} \tag{63}$$

and demonstrate that P_S is a basis projection on $\mathcal{H} \oplus \mathcal{H}$ with respect to $\Gamma \oplus (-\Gamma)$. Then realize ω as the Fock state on $\mathcal{A}_{\mathrm{CAR}}(P_S\mathcal{H} \oplus \mathcal{H})$.