Mathematical Statistical Physics, 2015
Homework Problems, LMU

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20. Consider the quasilocal UHF algebra $(\mathcal{A}, (\mathcal{A}_\Lambda)_{\Lambda \in \mathcal{F}(\mathbb{Z}^\nu)})$, $\nu \in \mathbb{N}$, associated with an infinite quantum spin system on $\mathbb{Z}^\nu$, where $\mathcal{F}(\mathbb{Z}^\nu)$ stands for the collection of finite subsets of $\mathbb{Z}^\nu$. We assume a self-adjoint interaction $\Phi : \mathcal{F}(\mathbb{Z}^\nu) \to \mathcal{A}$, $\Phi(\Lambda) = \Phi(\Lambda)^\star$, that is bounded in the sense that

$$||\Phi|| := \sup_{x \in \mathbb{Z}^\nu} \sum_{\Lambda \in \mathcal{F}(\mathbb{Z}^\nu), \Lambda \cap \{x\} \neq \emptyset} ||\Phi(\Lambda)|| \, < \, \infty. \tag{49}$$

Furthermore, Φ is of finite range, that is, there exists $R_\Phi \geq 1$ such that $\Phi(\Lambda) = 0$ if $R_\Phi < \operatorname{diam}(\Lambda) := \sup_{x,y \in \Lambda} |x - y|$. The local Hamiltonians are then defined by $H_\Lambda := \sum_{X \subset \Lambda} \Phi(X)$ for all $\Lambda \in \mathcal{F}(\mathbb{Z}^\nu)$.

(i) With $A \in \mathcal{A}_{\text{loc}} = \bigcup_{\Lambda \in \mathcal{F}(\mathbb{Z}^\nu)} \mathcal{A}_\Lambda$, prove that the limit

$$\delta(A) := \lim_{\Lambda \to \infty, \Lambda \in \mathcal{F}(\mathbb{Z}^\nu)} \mathrm{i}[H_\Lambda, A] \tag{50}$$

exists in $\mathcal{A}_{\text{loc}}$ and that $\delta(A)$ is a symmetric derivation with domain $\mathcal{D}(\delta) = \mathcal{A}_{\text{loc}}$ and $\delta(\mathcal{D}(\delta)) \subset \mathcal{D}(\delta)$. Here, the limit $\Lambda \to \infty$ is understood in the following way: For every sequence $\Lambda_1 \subset \Lambda_2 \subset \cdots$ in $\mathcal{F}(\mathbb{Z}^\nu)$ with $\cup_{n=1}^\infty \Lambda_n = \mathbb{Z}^\nu$ the limit $\lim_{n \to \infty} \mathrm{i}[H_{\Lambda_n}, A]$ exists and does not dependent on the choice of the sequence.

(ii) Show that for every $A \in \mathcal{A}_{\text{loc}}$ and for sufficiently small $t \in \mathbb{R}$ the series

$$e^{t\delta}(A) = \sum_{n=1}^\infty \frac{t^n}{n!} \delta^n(A) \tag{51}$$

converges in norm in $\mathcal{A}$ and therefore the function $t \mapsto e^{t\delta}(A)$ is analytic.
Hint: Argue that $\delta^n(A) \in \mathcal{A}_{\rm loc}$ and control $||\delta^n(A)|| \le c(n, ||\Phi||)||A||$ by estimating properly all commutators; you may use the inequality $a^n \le n!\, b^{-n} e^{ab}$ for $a, b > 0$ and $n \in \mathbb{N}$.

21. Consider now the quasilocal UHF algebra $(\mathcal{A}, (\mathcal{A}_\Lambda)_{\Lambda \in \mathcal{F}(\mathbb{Z}^\nu)})$ for $\nu = 1$ and associated with an infinite quantum spin system on $\mathbb{Z}$. Again $\mathcal{F}(\mathbb{Z})$ stands for the collection of finite subsets of $\mathbb{Z}$ and the single-site Hilbert space is given by $\mathcal{H}_x = \mathbb{C}^n$ with $n \in \mathbb{N}$. The interaction $\Phi : \mathcal{F}(\mathbb{Z}) \to \mathcal{A}$ is assumed to be hermitian, $\Phi(X) = \Phi(X)^\star$ with $\Phi(X) \in \mathcal{A}_X$ for all $X \in \mathcal{F}(\mathbb{Z})$. Also, Φ is assumed to be bounded and decaying at infinity,

$$\sup_{x \in \mathbb{Z}} \sum_{X \in \mathcal{F}(\mathbb{Z}^\nu), X \cap \{x\} \neq \emptyset} \mathrm{diam}(X) ||\Phi(X)|| < \infty. \tag{52}$$

In addition, Φ is invariant under the automorphism α, viz., $\alpha(\Phi(X)) = \Phi(X)$ for all $X \in \mathcal{F}(\mathbb{Z})$, where the automorphism α is defined locally by

$$\alpha(A) = \big(\otimes_{x \in \Lambda} U_x^\star\big) A \big(\otimes_{x \in \Lambda} U_x\big) \tag{53}$$

and extended to $\mathcal{A}$ by density. Here, each U_x is a unitary operator on $\mathcal{H}_x$. As in Problem 20, a strongly continuous one-parameter group τ_t, $t \in \mathbb{R}$ of $\star$ automorphisms of $\mathcal{A}$ is constructed as the limit of the local dynamics generated by the local Hamiltonians $H_\Lambda = \sum_{X \subset \Lambda} \Phi(X)$. This τ_t satisfies $\tau_t \circ \alpha = \alpha \circ \tau_t$ for all $t \in \mathbb{R}$.

Demonstrate that if ω is a (τ_t, β) KMS state for some $\beta \in (0, \infty)$, then $\omega \circ \alpha = \omega$ on $\mathcal{A}$.
Hint: Examine whether a certain theorem discussed in class can be applied.

22 Let $\mathcal{H}$ be a finite dimensional Hilbert space, $H = H^* \in \mathcal{B}(\mathcal{H})$ be the Hamiltonian generating the dynamics and $\rho_\beta = \mathrm{Tr}(\exp(-\beta H))^{-1} \exp(-\beta H)$ be the density matrix of the Gibbs state. For any $A \in \mathcal{B}(\mathcal{H})$, let $\gamma_t : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be defined for $t \ge 0$ by

$$\gamma_t(x) = \exp(tL_A)(x), \qquad L_A(x) = A^* x A - \frac{1}{2}(A^* A x + x A^* A). \tag{54}$$

Note that $L_A(1) = 0$ and that $\mathrm{Tr}(L_A(x) y) = \mathrm{Tr}(x L_{A^*}(y))$.

(i) Check that $\gamma_t(1) = 1$ and that $\gamma_t(x) > 0$ if $x > 0$

(ii) Prove that for any density matrix ρ, there is a unique density matrix ρ_t (namely positive and normalised) such that

$$\mathrm{Tr}(\rho_t x) = \mathrm{Tr}(\rho \gamma_t(x)) \tag{55}$$

(iii) Recall that the free energy functional if given by

$$F_\beta(\rho) = \mathrm{Tr}(\rho H) - \beta^{-1} S(\rho), \qquad S(\rho) = -\mathrm{Tr}(\rho \log \rho). \tag{56}$$

Use the variational principle applied to the family $\rho_{\beta,t}$, namely $F_\beta(\rho_{\beta,t}) \geq F_\beta(\rho_\beta)$ for all $t \in [0,1)$ to derive the energy-entropy balance inequality *Hint:* It suffices to consider $\lim_{t\to 0^+} t^{-1}[F_\beta(\rho_{\beta,t}) - F_\beta(\rho_\beta)]$. In order to bound the entropy term, you can use the joint convexity of $(u,v) \mapsto s(u\|v) := u \ln(u/v)$:

$$s\Big(\sum_i \lambda_i u_i \Big\| \sum_i \lambda_i v_i\Big) \leq \sum_i \lambda_i s(u_i\|v_i) \tag{57}$$

(iv) Reciprocally, prove that the unique $\rho > 0$ satisfying the energy-entropy balance inequality is the Gibbs state.
Hint: First, observe that if ρ satisfies the EEB inequality then $[\rho, H] = 0$. Secondly, use the observables $P_{ij} = |\psi_i\rangle\langle\psi_j|$, where $\{\psi_i\}_{i=1}^{\dim \mathcal{H}}$ is an eigenbasis of H.