Mathematical Statistical Physics, 2015
Homework Problems, LMU

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19. Consider the two self-adjoint operators H and V on the finite dimensional Hilbert space $\mathcal{H}$. For $t \in \mathbb{R}$ let τ_t^V be the $\star$ automorphism of the algebra $\mathcal{B}(\mathcal{H})$ defined by

$$\tau_t^V(A) = e^{\mathrm{i}t(H+V)} A \, e^{-\mathrm{i}t(H+V)} \tag{38}$$

Similarly, for $\beta \in (0, \infty)$, the thermal state $\omega_{\beta,V}$ is defined by

$$\omega_{\beta,V}(A) = \mathrm{Tr}\{A e^{-\beta(H+V)}\}/\mathrm{Tr}\{e^{-\beta(H+V)}\}. \tag{39}$$

The goal of this exercise is to derive a rigorous expansion of $\omega_{\beta,V}$ with respect to V, viz.,

$$\omega_{\beta,V}(A) = \sum_{n=0}^{\infty} \nu_n(A) \quad \text{for} \quad \beta||V|| < \log(2) \tag{40}$$

by using the propagator in the interaction picture

$$\Gamma_t^V = e^{\mathrm{i}t(H+V)} e^{-\mathrm{i}tH}. \tag{41}$$

(i) First, verify the following basic relations:

$$\begin{aligned}
e^{\mathrm{i}t(H+V)} &= \Gamma_t^V e^{\mathrm{i}tH} \\
\tau_t^V(A)\Gamma_t^V &= \Gamma_t^V \tau_t^0(A) \\
(\Gamma_t^V)^{-1} &= (\Gamma_t^V)^\star = \tau_t^0(\Gamma_{-t}^V) \\
\Gamma_{t+s}^V &= \Gamma_s^V \tau_s^0(\Gamma_t^V) \\
\partial_t \Gamma_t^V &= \mathrm{i}\Gamma_t^V \tau_t^0(V), \quad \Gamma_0 = 1.
\end{aligned} \tag{42}$$

(ii) In particular, $\Gamma^V_{\mathrm{i}\beta}$ is well-defined for all $\beta \in \mathbb{R}$. Use this fact and the first relation in (42) to express $\omega_{\beta,V}$ in terms of $\omega_{\beta,0}$.

(iii) Since $z \mapsto \tau^V_z(A)$ is an entire function, show that the Dyson expansion converges uniformly on all compact subsets of $\mathbb{C}$:

$$\Gamma^V_t = 1 + \sum_{k=1}^{\infty} (\mathrm{i}t)^k \int_{0 \leq s_1 \leq \ldots s_k \leq 1} \tau^0_{ts_1}(V) \cdots \tau^0_{ts_k}(V)\, ds_1 \cdots ds_k. \tag{43}$$

(iv) Employ the Golden-Thomson inequality

$$\mathrm{Tr}\{e^{A+B}\} \leq \mathrm{Tr}\{e^A e^B\} \tag{44}$$

and the Duhamel formula, viz., for a differentiable matrix valued function $t \mapsto F(t) \in \mathrm{Mat}_{n,n}(\mathbb{C})$

$$\frac{de^F}{dt}(t) = \int_0^1 e^{sF(t)} \frac{dF}{dt}(t) e^{(1-s)F(t)}\, ds, \tag{45}$$

to prove: For V as above, $0 < \beta < \infty$, and $\alpha \in \mathbb{C}$,

$$|\omega_{\beta,0}(\Gamma^{\alpha V}_{\mathrm{i}\beta}) - 1| \leq \exp(|\alpha\beta|\, ||V||) - 1. \tag{46}$$

(v) Use the result of (ii) and the relations (43) and (46) to establish the expansion (40). In particular, show that

$$\begin{aligned} \nu_0(A) &= \omega_{\beta,0}(A) \\ \nu_1(A) &= -\beta(V,\, A - \omega_{\beta,0}(A))_\beta, \end{aligned} \tag{47}$$

where

$$\begin{aligned} (x,y)_\beta &:= \int_0^1 \omega_\beta\big(y\tau_{\mathrm{i}s\beta}(x)\big)\, ds \\ &= Z^{-1} \int_0^1 \mathrm{Tr}\{e^{-s\beta H} x e^{-(1-s)\beta H} y\}\, ds. \end{aligned} \tag{48}$$