# Mathematical Statistical Physics, 2015
# Homework Problems, LMU

Issued: May 20, 2015; deadline for handing in the solutions:
May 27, 2015, 10 pm (22:00)

16. Let $\mathcal{A}$ be a C$^\star$ algebra with a unit element, and $\tau_t,\ t \in \mathbb{R}$, a strongly continuous one-parameter group of $\star$ automorphisms of $\mathcal{A}$. For $\beta \in \mathbb{R}$ prove the equivalence of the following conditions (A) and (B):

(A) The state $\omega$ is a $(\tau_t, \beta)$ KMS state

(B) For all $A, B \in \mathcal{A}$ and functions $f$ that enjoy a Fourier transform $\hat{f} \in C_{\rm c}^\infty(\mathbb{R})$

$$\int_{-\infty}^{\infty} f(t)\omega(A\tau_t(B))\,dt = \int_{-\infty}^{\infty} f(t+\mathrm{i}\beta)\omega(\tau_t(B)A) \tag{34}$$

*Hint*: For the direction (A) to (B), first prove the assertion for an analytic element $B$, and then invoke density arguments. For the direction (B) to (A), you may use a sequence $\hat{f}_n \in C_{\rm c}^\infty(\mathbb{R})$ with $0 \le \hat{f}_n \le 1$, $\hat{f}_n(\lambda) = 1$ if $|\lambda| \le n$, and $\hat{f}_n(\lambda) = 0$ if $|\lambda| \ge n+1$.

You also need the Paley and Wiener theorem: A measurable function $F$ on $\mathbb{C}$ is the inverse Fourier transform of a function $\hat{F} \in C_{\rm c}^\infty$ with supp$F \subset [-R, R],\ R > 0$, iff $F$ is entire analytic and for each $n \in \mathbb{N}$ there exists a constant $C_n$ such that

$$|F(z)| \le C_n \frac{\exp(R\,|\mathrm{Im} z|)}{(1+|z|)^n}. \tag{35}$$

17. Let $\mathcal{A} = \overline{\cup_{\Lambda \in \mathbb{Z}^\nu} \mathcal{A}_\Lambda}$ be a quantum spin system, and for $\Lambda \in \mathcal{F}(\mathbb{Z}^\nu)$ let $H_\Lambda = H_\Lambda^\star \in \mathcal{A}_\Lambda$ be the corresponding finite volume Hamiltonian. Assume

that for $\Lambda_n = [-n,n]^\nu \cap \mathbb{Z}^\nu$ and all $A \in \mathcal{A}_{\rm loc}$, $t \in \mathbb{R}$, the Heisenberg dynamics

$$\tau_t^{\Lambda_n}(A) := \exp(\mathrm{i}tH_{\Lambda_n})\, A\, \exp(-\mathrm{i}tH_{\Lambda_n}) \tag{36}$$

has a limit $\tau_t(A)$ as $n \to \infty$.

(i) Prove that $\tau_t$ is a strongly continuous group of $\star$ automorphisms of $\mathcal{A}$. *Hint*: You may use that $\tau_t^{\Lambda_n}(A)$ is a strongly continuous group of $\star$ automorphisms of $\mathcal{A}_{\Lambda_n}$.

(ii) Show that $\mathcal{A}_{\rm loc} \subset \mathcal{D}(\delta)$, where $\delta$ is the generator of $\tau_t$.

(iii) Use the EEB inequality to prove the existence a $(\tau, \beta)$ KMS state for any $\beta > 0$.

18. Consider a C$^\star$ algebra $\mathcal{A}$ with a unit and a strongly continuous one-parameter group of $\star$ automorphisms of $\mathcal{A}$. Let $\omega$ be a $\tau_t$ invariant state such that

(i) $\tau_t$ is asymptotically abelian: $\lim_{t\to\infty} ||\,[A, \tau_t(B)]\,|| = 0$ for all $A, B \in \mathcal{A}$

(ii) $\omega$ has the "cluster property:" $\omega(A\tau_t(B)) \to \omega(A)\omega(B)$ holds as $t \to \infty$ for all $A, B \in \mathcal{A}$ (This is the typical property exhibited by KMS states with respect to an asymptotically abelian dynamics)

Prove that any $\omega$-normal state $\nu$ "returns to equilibrium" in the sense that upon $t \to \infty$

$$\nu(\tau_t(A)) \to \omega(A) \tag{37}$$

for all $A \in \mathcal{A}$.