Mathematical Statistical Physics, 2015
Homework Problems, LMU

Issued: May 13, 2015; deadline for handing in the solutions:
May 20, 2015, 10 pm (22:00)

13. (10 P) Let $\mathcal{A}_{\mathrm{CCR}}$ stand for the Weyl algebra over a given separable Hilbert space $\mathcal{H}$, and let $S : \mathcal{H} \to \mathcal{H}$ denote a real invertible linear map obeying $\mathrm{Im}(\langle Sf, Sg\rangle) = \mathrm{Im}(\langle f, g\rangle)$. The Bogoliubov $\star$ automorphism $\gamma : \mathcal{A}_{\mathrm{CCR}} \to \mathcal{A}_{\mathrm{CCR}}$ associated with S is defined by $\gamma(W(f)) := W(Sf)$ for all $f \in \mathcal{H}$. Furthermore, we denote by $(\mathcal{F}_+, \pi, \Omega)$ the Fock space representation of $\mathcal{A}_{\mathrm{CCR}}$ (recall that this representation is regular).

For any $f \in \mathcal{H}$, by

$$\begin{aligned} \exp\big(\mathrm{i}\tilde{\Phi}_\pi(f)\big) &:= \pi\big(\gamma(W(f))\big) \\ \tilde{a}_\pi(f) &:= \big(\tilde{\Phi}_\pi(f) + \mathrm{i}\tilde{\Phi}_\pi(\mathrm{i}f)\big)/\sqrt{2} \\ \tilde{a}^\star_\pi(f) &:= \big(\tilde{\Phi}_\pi(f) - \mathrm{i}\tilde{\Phi}_\pi(\mathrm{i}f)\big)/\sqrt{2} \end{aligned} \tag{22}$$

unbounded operators in $\mathcal{F}_+$ are defined; below, all domain question for these operators may be ignored and the necessary algebraic manipulations may be preformed on a formal level.

(i) Construct two maps $L : \mathcal{H} \to \mathcal{H}$ and $A : \mathcal{H} \to \mathcal{H}$ in terms of S such that

$$\begin{aligned} \tilde{a}_\pi(f) &= a(Lf) + a^\star(Af) \\ \tilde{a}^\star_\pi(f) &= a(Af) + a^\star(Lf) \end{aligned} \tag{23}$$

for all $f \in \mathcal{H}$. *Hint*: Employ the fact that $\pi : \mathcal{A}_{\mathrm{CCR}} \to \mathcal{B}(\mathcal{F}_+)$ is a regular representation to give a meaning to the generator of $t \mapsto \pi(W(tf))$, $t \in \mathbb{R}$, and compare it with $\tilde{\Phi}_\pi(f)$.

(ii) If $\tilde{a}_\pi(f)$ and $\tilde{a}^\star_\pi(f)$ are required to satisfy the CCR as operators on $\mathcal{F}_+$, show that

$$\begin{aligned} L^\star L - A^\star A = 1 = LL^\star - AA^\star \\ L^\star A - A^\star L = 0 = AL^\star - LA^\star. \end{aligned} \tag{24}$$

Hint: Use the canonical commutation relations on the one hand, and the invertibility of the map γ on the other hand.

(iii) Assume that $\mathrm{Tr} A^\star A < \infty$. Prove that

$$\langle \Omega,\, \tilde{N}_\pi \Omega \rangle_{\mathcal{F}} = \mathrm{Tr} A^\star A \tag{25}$$

where $\tilde{N}_\pi := \sum_{n=1}^\infty \tilde{a}^\star_\pi(f_n)\tilde{a}_\pi(f_n)$ and $(f_n)_{n\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}$.

Remark: Note that the Bogoliubov transformation changes the meaning of a “particle” and (25) shows that the number of particles with respect to the original ground state is bounded if A is Hilbert-Schmidt. In fact, one can prove that the above manipulations are well-defined if and only if A is Hilbert-Schmidt.

14. (5 P) The purpose of this exercise is to prove the uniqueness of the irreducible representation of the CAR algebra $\mathcal{A}_-(\mathbb{C})$, generated by $1, a$. Consider the self-adjoint elements

$$\psi_1 := a + a^*, \qquad \psi_2 := \mathrm{i}(a - a^*) \tag{26}$$

(i) Compute $\{\psi_i, \psi_j\}$, $i, j = 1, 2$

(ii) Use the Pauli matrices to exhibit a representation of $\mathcal{A}_-(\mathbb{C})$ in $\mathbb{C}^2$ and show that it is irreducible.

(iii) Let ϕ_1, ϕ_2 be an arbitrary representation of ψ_1, ψ_2 in a Hilbert space $\mathcal{H}$. Determine the spectrum of $\phi_0 := \mathrm{i}\phi_1\phi_2$

(iv) Let $\mathcal{K} := \mathrm{Ker}(\phi_0 - 1)$. Prove that the map

$$\mathcal{H} \ni \xi \longmapsto U\xi := \left(\frac{\phi_1}{2}(1 - \phi_0)\xi, \frac{1}{2}(1 + \phi_0)\xi \right) \tag{27}$$

is a unitary map $\mathcal{H} \to \mathcal{K} \oplus \mathcal{K}$.

(v) Show that

$$U\phi_1 = (\sigma^1 \otimes 1_{\mathcal{K}})U, \qquad U\phi_2 = (\sigma^2 \otimes 1_{\mathcal{K}})U \tag{28}$$

(vi) Conclude that the representation on $\mathcal{H}$ is irreducible if and only if $\mathcal{K} = \mathbb{C}$.

15. (15 P) Consider the Hilbert space $\mathcal{H} = \mathbb{C}^n$ and the C* algebra $\mathcal{A} = \mathrm{Mat}_{n,n}(\mathbb{C})$ with $n \in \mathbb{N}$.

(i) Take any positive $A, B > 0,\ A, B \in \mathcal{A}$, and any convex $f \in C^1((0,\infty),\mathbb{R})$. Show that

$$\mathrm{Tr}\{f(A) - f(B) - (A - B)f'(B)\} \geq 0 \tag{29}$$

and that for a strictly convex f equality holds iff $A = B$.

(ii) If $\beta > 0$, H self-adjoint Hamiltonian, and ω a state on $\mathcal{A}$, we set

$$F_\beta(\omega) := -\beta^{-1} S(\omega) + \omega(H) \tag{30}$$

where $S(\omega) = -\mathrm{Tr}\{\rho_\omega \log \rho_\omega\}$ denotes the entropy of ω. Establish the existence of a unique minimizer $\omega_{\rho_{\beta H}}$ with

$$F_\beta(\omega_{\rho_{\beta H}}) = \inf\{F_\beta(\omega) \,|\, \omega \text{ is a state on } \mathcal{A}\} \tag{31}$$

and show that $\omega_{\rho_{\beta H}}$ is given by the Gibbs state at inverse temperature β associated to

$$\rho_{\beta H} = \frac{e^{-\beta H}}{\mathrm{Tr}\{e^{-\beta H}\}} \tag{32}$$

and compute $F_\beta(\omega_{\rho_{\beta H}})$. *Hint*: Demonstrate that

$$F_\beta(\omega_\rho) = \beta^{-1} \log \mathrm{Tr}\{e^{-\beta H}\} + \beta^{-1} \mathrm{Tr}\{\rho \log \rho - \rho \log \rho_{\beta H}\} \tag{33}$$

and use (i) with the choice $f(t) = t \log t$.

(iii) Prove that $t \mapsto \tau_t$ is strongly continuous.

(iv) For a self-adjoint $H \in \mathcal{A}$ consider the one-parameter group $\{\tau_t \,|\, t \in \mathbb{R}\}$ of $\star$ automorphisms $A \mapsto \tau_t(A) := \exp(\mathrm{i}tH)\, A \exp(-\mathrm{i}tH)$ of $\mathcal{A}$. If $\beta > 0$, prove that a state ω on $\mathcal{A}$ is a (τ_t, β) KMS state iff ω equals the Gibbs state at inverse temperature β, viz., $\omega = \omega_{\rho_{\beta H}}$. Also prove that $t \mapsto \tau_t$ is strongly continuous.

(v) Assume $\beta = 1$ and let ω be a state on $\mathcal{A}$. Give a necessary and sufficient condition for the existence of a dynamics σ_t for which ω is a $(\sigma, 1)$-KMS state. Is σ generated by a Hamiltonian K? Are σ and K unique?