Mathematical Statistical Physics, 2015
Homework Problems, LMU

Issued: April 29, 2015; deadline for handing in the solutions:
May 6, 2015, 10 pm (22:00)

7. Let $\mathcal{A}$ be the C* algebra $\mathrm{Mat}_{2,2}$ of 2×2 complex matrices with its usual linear and algebraic structure and operator norm $||\cdot||_{\mathrm{op}}$, and consider the density matrix

$$\rho_\alpha = \begin{pmatrix} 1-\alpha & 0 \\ 0 & \alpha \end{pmatrix} \tag{11}$$

that depends on the parameter $\alpha \in [0, 1/2]$. Construct explicitly the GNS representation $(\mathcal{H}_\alpha, \pi_\alpha, \Omega_\alpha)$ of $\mathcal{A}$ associated with the state ω_α, where ω_α is the state induced by ρ_α, in case of (i) $\alpha = 0$; (ii) $\alpha = 1/4$; ; (iii) $\alpha = 1/2$. Determine whether ω_α is pure or mixed, and whether or not it is a vector state for the respective representation.

8. Consider the set of continuous functions $C([0,1])$ with the norm $||f|| = \sup\{|f(x)|\,|0 \le x \le 1\}$ as a C* algebra $\mathcal{A}$.

(i) Prove that by $\omega(f) := \int_0^1 f(x)\,dx$ for all $f \in \mathcal{A}$ a state is defined on $\mathcal{A}$.

(ii) Construct the GNS representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ associated with this state ω.

(iii) Show the equality $||\pi_\omega(f)|| = ||f||$ for all $f \in \mathcal{A}$.

9. A normalized positive linear functional ω on a C* algebra $\mathcal{A}$ is called a "tracial state" iff $\omega(a^\star a) = \omega(a a^\star)$ for all $a \in \mathcal{A}$.

(i) Show that ω being a tracial state on $\mathcal{A}$ is equivalent to: ω is a normalized positive linear functional ω on $\mathcal{A}$ and $\omega(ab) = \omega(ba)$ for all $a, b \in \mathcal{A}$.

(ii) If $\mathcal{A}$ is given by the C* algebra $\mathrm{Mat}_{n,n}$ (discussed, e.g., in Problem 6), prove that there exists a tracial state ω on $\mathcal{A}$ and that this ω is unique.

(iii) Consider the C* algebra of compact operators $\mathcal{J}_\infty \subset \mathcal{B}(\mathcal{H})$ on the Hilbert space $\mathcal{H}$. Demonstrate that there does not exist a tracial state ω on $\mathcal{A}$ if $\dim \mathcal{H} = \infty$.

Hint: Examine ω for finite dimensional subalgebras of $\mathcal{A}$.