Mathematical Statistical Physics, 2015
Homework Problems, LMU

Issued: April 22, 2015; deadline for handing in the solutions:
April 29, 2015, 10 pm (22:00)

4. If $\mathcal{A}$ is a C* algebra without unit element, show that by setting $\mathcal{A}_{\rm I} = \mathcal{A} \oplus \mathbb{C}$ (as linear spaces) and introducing the product $(a \oplus \alpha)(b \oplus \beta) = (ab+\alpha b+\beta a, \alpha\beta)$, the involution $(a\oplus\alpha)^\star = a^\star \oplus \overline{\alpha}$, and the norm $||a\oplus\alpha||_{\mathcal{A}_{\rm I}} = \sup\{||ab+\alpha b||_{\mathcal{A}} \,|\, b \in \mathcal{A}, ||b||_{\mathcal{A}} \leq 1\}$ (for all $\alpha, \beta \in \mathbb{C}$, $a, b \in \mathcal{A}$), a C* algebra with unit $\mathrm{I} = 0 \oplus 1$ is defined. In particular, check that by $||\cdot||_{\mathcal{A}_{\rm I}}$ indeed a norm on $\mathcal{A}_{\rm I}$ is provided. (Remark: This construction of $\mathcal{A}_{\rm I}$ implies that $\mathcal{A}$ is isometrically $\star$ isomorphic to a C* subalgebra of codimension one of $\mathcal{A}_{\rm I}$).

5. Consider a C* algebra $\mathcal{A}$ with unit element I, and the spectrum $\sigma(a) = \{\lambda \in \mathbb{C} \,|\, \lambda\mathrm{I} - a \text{ has no inverse in } \mathcal{A}\}$ of a normal element $a \in \mathcal{A}$. Prove that for all $a \in \mathcal{A}$ the limit $\lim_{n\to\infty} ||a^n||^{1/n}$ exists and equals the spectral radius $r(a) := \sup_{\lambda\in\sigma(a)} |\lambda|$ of a. In addition, show that $r(a) = ||a||$.
Hint: Prove the complementary inequalities $r(a) \leq \liminf_n ||a^n||^{1/n}$ and $r(a) \geq \limsup_n ||a^n||^{1/n}$. To derive the first inequality, study the series $\lambda^{-1}\sum_k (a/\lambda)^k$; for the second inequality, investigate the radius of convergence of this series. To establish $r(a) = ||a||$, consider $||a^{2^k}||^2$.

6. Starting with the set $\mathrm{Mat}_{n,n}$ of $n \times n$ complex matrices with its usual vector space structure, this becomes an associative algebra if the standard matrix multiplication serves as the algebra multiplication. Banach spaces $(\mathrm{Mat}_{n,n}, |, ||\cdot||_{\rm op})$ and $(\mathrm{Mat}_{n,n}, \, ||\cdot||_{\rm HS})$ are obtained by choosing the norms $||A||_{\rm op} = \sup\{||Ax|| \;|\, x \in \mathbb{C}^n, ||x|| = 1\}$ (with $||x|| = (\sum_{i=1}^n |x_i|^2)^{1/2}$ for $x \in \mathbb{C}^n$) and $||A||_{\rm HS} = (\mathrm{Tr} A^\star A)^{1/2}$.

(i) Does the involution defined by $A^\star :=$ adjoint of A render the algebras $(\mathrm{Mat}_{n,n}, ||\cdot||_{\mathrm{op}})$ and $(\mathrm{Mat}_{n,n}, ||\cdot||_{\mathrm{HS}})$ into C* algebras?

(ii) Consider the linear space of matrix valued continuous functions $\mathcal{A} := \{f : [-1,1] \to \mathrm{Mat}_{2,2}(\mathbb{C}) \,|\, f(t) \text{ is diagonal if } |t| = 1\}$ with the norm $||f|| = \sup_{t\in[-1,1]} ||f(t)||_{\mathrm{op}}$ and matrix multiplication and involution $f \mapsto f^\star$, $f^\star(t) := f(t)^\star$. Is this $\mathcal{A}$ a C* algebra?