Mathematical Statistical Physics, 2015
Homework Problems, LMU

Issued: June 24, 2015; deadline for handing in the solutions:
July 1, 2015, 10 pm (22:00)

28. Consider the function $f : \mathbb{R}^4 \to \mathbb{R}$ defined by $f(x) = |x|^{-4}$. Since $f \notin L^1_{\mathrm{loc}}(\mathbb{R}^4)$, the function f does not lead to a distribution T_f on the space of test functions $\mathcal{D}(\mathbb{R}^4)$ in the standard way. On the other hand, taking the subspace $\mathcal{D}_0(\mathbb{R}^4) := \{g \in \mathcal{D}(\mathbb{R}^4) \,|\, g(0) = 0\}$, we can define $T_f : \mathcal{D}_0 \to \mathbb{C}$ by $T_f(\phi) := \int_{\mathbb{R}^4} f(x)\phi(x)\,dx$.

(i) Show that T_f is a linear and well-defined map on $\mathcal{D}_0$, viz., $T_f(\phi) < \infty$. (T_f is also continuous in the topology of distributions.)

(ii) Consider $w \in \mathcal{D}(\mathbb{R}^4)$ with $w(0) = 1$ and the map $\tilde{T}_f^{(w)} : \mathcal{D} \to \mathbb{C}$ given by

$$\tilde{T}_f^{(w)}(\phi) := \int_{\mathbb{R}^4} f(x)\big(\phi(x) - w(x)\phi(0)\big)\,dx. \tag{71}$$

Prove that $\tilde{T}_f^{(w)}$ is a distribution, $\tilde{T}_f^{(w)} \in \mathcal{D}'(\mathbb{R}^4)$, and that $\tilde{T}_f^{(w)}$ extends T_f, i.e., $\tilde{T}_f^{(w)}(\phi) = T_f(\phi)$ if $\phi \in \mathcal{D}_0(\mathbb{R}^4)$.

(iii) For $\lambda > 0$, let $D_\lambda : \mathcal{D}_0(\mathbb{R}^4) \to \mathcal{D}_0(\mathbb{R}^4)$ denote the scaling $(D_\lambda\phi)(x) = \phi(\lambda x)$. Prove the scale invariance of $\mathcal{D}_0(\mathbb{R}^4)$ and of T_f, i.e., $T_f \circ D_\lambda(\phi) = T_f(\phi)$ for all $\phi \in \mathcal{D}_0(\mathbb{R}^4)$. Also compute the distribution $\tilde{T}_f^{(w)} \circ D_\lambda - \tilde{T}_f^{(w)}$.

29. Let $H : C^\infty([0, 2\pi] \times \mathbb{R}) \to C^\infty(\mathbb{R})$ be the classical field Hamiltonian given by

$$H(p) = \int_0^\pi \left(\left(\frac{\partial p}{\partial x}\right)^2 + \left(\frac{\partial p}{\partial t}\right)^2\right) dx \tag{72}$$

where $p(0,t) = p(\pi,t) = 0$.

(i) Employ the Fourier expansion of $p(\cdot,t)$ to express H as an infinite series $H = \sum_{k=1}^{\infty} h_k$ of classical harmonic oscillators of mass 1/2

$$h_k = \left(\frac{\partial p_k}{\partial t}\right)^2 + \omega_k p_k^2 \tag{73}$$

for appropriate frequencies ω_k.

(iii) Quantize the Hamiltonians h_k by the rules $p_k \leftrightarrow x_k$ and $\partial p_k/\partial t \leftrightarrow -\mathrm{i}\partial/\partial x_k$, where the expressions on the right hand sides are the multiplication and momentum operators, respectively, in $L^2(\mathbb{R})$. Show that $\sum_{k=1}^{n} E_k \to \infty$ as $n \to \infty$ for the ground state energies E_k of the obtained harmonic oscillator operators $\hat{h}_k$.

(iiii) To avoid the divergence found in (ii), we assume that H can be modified to a Hamiltonian H_{reg} such that every harmonic oscillator $\hat{h}_k$ has a ground state energy $e^{-a/\lambda_k} E_k$; here $a > 0$ stands for a reference distance and λ_k denotes the wave length of the mode k (so that this affects mainly the modes with $\lambda_k << a$). Prove the asymptotics

$$\sum_{k=1}^{\infty} E_k e^{-a/\lambda_k} = a^{-2} - 1/12 + O(a) \tag{74}$$

as $a \downarrow 0$.

(iv) As a consequence of (iii), by adding a p independent and locality preserving contribution to H_{reg} to obtain a renormalized Hamiltonian

$$H' = H_{\text{reg}} - \text{const} \int_0^{\pi} a^{-2}\, dx, \tag{75}$$

the ground state energy of H' can be made finite for $a \downarrow 0$. Discuss the value of this ground state energy as compared to the value $\zeta(-1)$ of the Riemann zeta function ζ.
Remark: The form of H' is motivated by
Locality, viz., the form of the Hamiltonian should be the space integral of a Hamiltonian density which generally is a function of $p(\cdot,t)$ and its derivatives at one point only;

Dimensional analysis, viz., upon a scaling $(x,t) \mapsto (zx, zt)$, the classical Hamiltonian is transformed to $z^{-1}H$ for $z > 0$. Similarly λ_k scales like $z\lambda_k$ and thus we also need to scale a as a length, that is, like za. As a result, the only Hamilton density $h(a)$ with $\int h(a)dx \mapsto z^{-1} \int h(a)dx$ is given by $h(a) = \text{const}\, a^{-2}$.

30. Consider the two-dimensional Ising model and compute the effective Hamiltonian after one block spin transformation that is accomplished by taking the sum over all spins S_{ij} with $i + j$ even.
Hint: First show that the ansatz

$$\exp\big(J(S_a + S_b + S_c + S_d)\big) + \exp\big(-J(S_a + S_b + S_c + S_d)\big) = \\ f \exp\big(K_{\text{nn}}(S_aS_b + S_bS_c + S_cS_d + S_dS_a) + K_{\text{nnn}}(S_aS_c + S_bS_d) + K_q S_aS_bS_cS_d\big) \tag{76}$$

can be used to determine the numbers $f, K_{\text{nn}}, K_{\text{nnn}}, K_q$ such that (76) holds for all assignments $S_a, S_b, S_c, S_d \in \{-1, 1\}$.