# Mathematical Statistical Physics, 2015
# Homework Problems, LMU

Issued: June 17, 2015; deadline for handing in the solutions: June 24, 2015, 10 pm (22:00)

26. Let $\Lambda \subset \mathbb{R}^\nu$ be a connected bounded open region of $\mathbb{R}^\nu$, $3 \leq \nu \in \mathbb{N}$, and define $\Lambda_L = \{x \in \mathbb{R} \,|\, x/L \in \Lambda\}$ for any $L > 0$. Let $h_L = -\Delta$ be the self-adjoint Hamiltonian with Dirichlet boundary conditions on $\partial\Lambda_L$, i.e., the closure of the Laplacian on $\{f \in C^\infty(\overline{\Lambda_L}) \,|\, f_{|\partial\Lambda_L} = 0\}$ where the boundary $\partial\Lambda_L$ is assumed to be sufficiently smooth. Then $h_L$ enjoys a purely discrete spectrum, and the asymptotic distribution of its eigenvalues $\lambda_i$ obeys the Weyl law,

$$\lim_{\lambda\to\infty} \lambda^{-\nu/2} N(\lambda) = \text{const.} \tag{64}$$

Here, $N(\lambda)$ stands for the number of eigenvalues of $h_L$ that do not exceed $\lambda$, and the constant depends on the volume of $\lambda_L$. For $\beta > 0$ and $0 < z \leq 1$ we define

$$\rho_L(z,\beta) = \frac{1}{|\lambda_L|} \operatorname{Tr} \frac{z\exp(-\beta h_L)}{1 - z\exp(-\beta h_L)}. \tag{65}$$

Assuming $\lambda_1 < \lambda_2$, for $\overline{\rho} > 0$, let $z_L := \exp(\beta\mu_L)$ be the unique solution to

$$\overline{\rho} = \rho_L(z_L, \beta) \tag{66}$$

with $\mu_L < h_L$. Employing the expansion

$$\rho_L(z_L,\beta) = \sum_{n=1}^{\infty} \rho_L^{(n)}(z_L,\beta) \tag{67}$$

with

$$\rho_L^{(n)}(z_L,\beta) = \frac{1}{|\Lambda_L|} \langle \psi_L^{(n)}, \frac{z_L\exp(-\beta h_L)}{1 - z_L\exp(-\beta h_L)} \psi_L^{(n)} \rangle \tag{68}$$

and $\psi_L^{(n)}$ being the $n^{\text{th}}$ eigenfunction of $h_L$, $n = 1, 2, \ldots$, prove that for $n > 1$

$$\lim_{L\to\infty} \rho_L^{(n)}(z_L, \beta) = 0. \tag{69}$$

27. For the system discussed in problem 26, show that in the limit

$$\lim_{L\to\infty} \sum_{n=2}^{\infty} \rho_L^{(n)}(z_L, \beta) = \text{const}, \tag{70}$$

where the constant is finite and independent of $\overline{\rho}$. (In class it was claimed that this constant is given by $\rho_{\text{c}}(\beta)$ and therefore $\lim_{L\to\infty} \rho_L^{(1)}(z_L, \beta) > 0$).