Mathematical Statistical Physics, 2015
Homework Problems, LMU

Issued: April 15, 2015; deadline for handing in the solutions:
April 22, 2015, 22 pm

1.Consider a lattice $\Lambda \subset \mathbb{Z}^\nu$, $\nu \in \mathbb{N}$, of volume $N := |\Lambda|$ with periodic boundary conditions. The set of configurations is given by $\Omega = \{\pm 1\}^\Lambda$, and the energy of a configuration $s \in \Omega$ is

$$H(s) = -\frac{1}{2}\sum_{i,k} J_{ik} s_i s_k - h \sum_i s_i, \qquad J_{ik} \geq 0. \tag{1}$$

The *mean-field approximation* is defined by replacing s_k by the average magnetisation

$$m(s) := \frac{1}{N}\sum_k s_k, \tag{2}$$

while the couplings J_i of s_i to the mean field is constant, yielding

$$H_N^{\mathrm{MF}}(s) = -N\left[\frac{1}{2}Jm(s)^2 + hm(s)\right]. \tag{3}$$

(i) Show that the canonical partition function

$$Z^{\mathrm{MF}}(\beta, h, N) := \sum_s \mathrm{e}^{-\beta H_N^{\mathrm{MF}}(s)} = \sum_{m \in M_N} \mathrm{e}^{-N\beta f(m) + o(N)} \tag{4}$$

where M_N is the set of possible values of $m(s)$, and

$$f(m) := -[(J/2)m^2 + hm] + \beta^{-1}\left[\frac{1+m}{2}\ln\frac{1+m}{2} + \frac{1-m}{2}\ln\frac{1-m}{2}\right] \tag{5}$$

is the mean-field *free energy functional*.

Hint: Count the number of spin configurations corresponding to each $m \in M_N$ and use Stirling's formula.

(ii) Equilibrium states are given by the minimisers of $f(m), -1 \leq m \leq 1$. Show that they satisfy the transcendental equation

$$m = \tanh \beta(Jm + h) \tag{6}$$

(iii) Defining the critical temperature β_c by the smallest β for which more than one equilibrium state exist, and letting $h = 0$, show that the critical temperature is given by

$$\beta_c = J^{-1}. \tag{7}$$

(iv) Introduce $x := \beta(Jm + h)$. Discuss the uniqueness of equilibrium states in the two cases $\beta J \leq 1$ (high temperature) and $\beta J \geq 1$ (low temperature), for various cases of h

Note that the above results are dimension independent.

2. Show that there is a symmetry breaking in the three-dimensional Ising model when the temperature drops sufficiently low. You can proceed in close analogy to the two-dimensional case discussed in the lecture. At some point, you will have to count surfaces enclosing the origin. This is best done in a step-wise procedure, where in each step you glue a square to each boundary edge of the (open) surface produced in the previous step

3. Let $\mathrm{Mat}_{n,n}$ stand for the set of $n \times n$ matrices with complex elements. Two matrices $A, B \in \mathrm{Mat}_{n,n}$ are called *similar* ($A \sim B$) iff there is $S \in GL(n, \mathbb{C})$ such that $A = SBS^{-1}$. This is an equivalence relation and we denote the set of equivalence classes by $M := \mathrm{Mat}_{n,n}/\sim$ (In this context, recall the concept of the Jordan normal form of a matrix).

(i) Show that for $k \in \mathbb{N}$

$$T_k \colon M \to \mathbb{C} \qquad T_k([A]) := \mathrm{tr}(A^k) \tag{8}$$

is well defined (i.e. is independent of the representative A of the equivalence class $[A]$).

(For enthusiasts: Use the Cayley-Hamilton Theorem to show that for $k > n$, one can express T_k as a polynomial in $T_1, \dots T_n$. (This justifies viewing $T_1, \dots, T_n$ as "coordinates" on M.))

(ii) Give an example of $[A_1], [A_2] \in M$ with

$$T_k([A_1]) = T_k([A_2]) \quad \text{for all } k \tag{9}$$

but $[A_1] \neq [A_2]$. ("Traces do not separate points.")

(iii) Prove that there is no continuous injection $f\colon M \to \mathbb{C}$ with $f([A_1]) \neq f([A_2])$ for $[A_1], [A_2]$ obeying (9) ("There are no further coordinates on M.").

Hint: Provide a continuous $S\colon (0,1) \to GL(n, \mathbb{C})$ with

$$\lim_{\epsilon \to 0} S(\epsilon) A_1 S(\epsilon)^{-1} = A_2. \tag{10}$$