Mathematical Quantum Mechanics, 2014/15 Homework Problems, LMU

Issued: November 11, 2014; deadline for handing in the solutions: November 18, 2014, 4 pm

17. Demonstrate that at $t = 0$

$$\frac{d}{dt}\int \big(\rho(x) + tg(x)\big)^{5/3} dx = \int \frac{d}{dt}\big(\rho(x) + tg(x)\big)^{5/3} dx \tag{19}$$

for the densities ρ and g discussed in the lecture, i.e., $\operatorname{supp} g \subset \{x \,|\, \rho(x) > \delta > 0\}$ and $-\delta/2 < t < \delta/2$.

18. Let $\Omega \subset \mathbb{R}^3$ be a bounded region, $\rho \in L^2(\Omega)$. Show that the equation $-\Delta\phi = 4\pi\rho$ in Ω with $\phi = 0$ on $\partial\Omega$ enjoys a unique (weak) solution in the local Sobolev space $H_0^1(\Omega)$. (Hint: Study the functional $p(\phi) = \int_\Omega |\nabla\phi|^2 - 8\pi\int\rho\phi$ and recall the Poincaré inequality to infer the existence of a finite infimum of p.)

19. Consider the Thomas-Fermi functional for an atom, viz., $\mathcal{E}_{\mathrm{TF}}(\rho) = (3/5)\gamma_{\mathrm{TF}}\int\rho(x)^{5/3}d^3x - \int Z|x|^{-1}\rho(x)d^3x + D(\rho,\rho)$.

(i) Show that the minimizing ρ of $\mathcal{E}_{\mathrm{TF}}$ is spherically symmetric.

(ii) Prove (without using the Thomas-Fermi equation) that $\int\rho(x)d^3x > Z$ cannot hold for the minimizer ρ of $\mathcal{E}_{\mathrm{TF}}$. (Hint: Suppose $\int\rho(x)d^3x > Z$ and then take the smallest $R > 0$ such that $\int_{|x|<R}\rho(x)d^3x = Z$ and estimate the difference $\mathcal{E}_{\mathrm{TF}}(\rho_R) - \mathcal{E}_{\mathrm{TF}}(\rho)$ for ρ_R defined by $\rho_R(x) = \rho(x)$ if $|x| \le R$, and $\rho_R(x) = 0$ otherwise.