Mathematical Quantum Mechanics, 2014/15
Homework Problems, LMU

Issued: October 28, 2014; deadline for handing in the solutions: November 04, 2014, 4 pm

10. Provide a counterexample demonstrating that $0 \leq A \leq B$ for two operators A, B on a finite-dimensional Hilbert space does not necessarily imply that $A^2 \leq B^2$.

11. Assuming an electrostatic potential V such that the Dirac operator $D_V = D_0 + V$ is self-adjoint on $\mathcal{D}(D_V) = H^1(\mathbb{R}^3, \mathbb{C}^4) \subset \mathfrak{H} = L^2(\mathbb{R}^3, \mathbb{C}^4)$. Using the conjugation operator $\tau : \mathfrak{H} \to \mathfrak{H}$ with $\tau\psi := \overline{\psi}$, we construct the operator $C : \mathfrak{H} \to \mathfrak{H}$ ("Kramers' operator") by

$$C = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} \tau. \tag{12}$$

(i) Prove that $[D_V, C] = 0$.

(ii) What does the vanishing of the commutator in (i) imply for the degeneracy of the eigenvalues of D_V, i.e., what is their minimal degeneracy?

12. Consider the functional discussed in the lecture in context of the Brown-Ravenhall operator (cf., W. D. Evans, P. Perry, and H. Siedentop, Commun. Math. Phys. 178 (1996) 733-746)

$$e[a] = \int_0^\infty p|a(p)|^2 - \frac{\gamma}{2\pi}\int_0^\infty\int_0^\infty dq dp\, \overline{a}(q)a(p)\big(Q_0(\frac{1}{2}(\frac{p}{q}+\frac{q}{p}))+Q_1(\frac{1}{2}(\frac{p}{q}+\frac{q}{p}))\big), \tag{13}$$

where $Q_\ell(z) = (1/2)\int_{-1}^1 (z-t)^{-1} P_\ell(t)\, dt$ and P_ℓ denote the Legendre polynomials.

(i) Show that $e[a]$ is bounded below for all $a \in C_0^\infty(\mathbb{R})$ if $\gamma \leq \gamma_{\text{cri}} = 2/(\pi/2 + 2/\pi)$.

(ii) Show that $e[a]$ is no longer bounded below if $\gamma > \gamma_{\text{cri}}$.