Mathematical Quantum Mechanics, 2014/15 Homework Problems, LMU

Issued: October 21, 2014; deadline for handing in the solutions: October 28, 2014, 4 pm

6. Let us consider the Hilbert space $\mathcal{H} = L^2(0,1)$ and the momentum operator for a particle moving on a circle, i.e., $p_\mathrm{c} = -\mathrm{i}d/dx$ with domain $\mathcal{D}(p_\mathrm{c}) = \{\psi \in \mathcal{H} \,|\, \psi \in \mathrm{AC}(0,1), \psi' \in \mathcal{H}, \psi(0) = \psi(1)\}$. Determine the resolvent set $\rho(p_\mathrm{c})$, and for $z \in \rho(p_\mathrm{c})$ compute the resolvent $(p_\mathrm{c} - z)^{-1}$ of this operator.

7. Starting with the kinetic energy operator $H_0 = -\Delta$ on the Sobolev space $\mathcal{D}(H_0) = H^2(\mathbb{R}^3) \subset \mathcal{H} = L^2(\mathbb{R}^3)$, and $a \in \mathbb{R}^3 \setminus \{0\}$, are the following operators bounded relative to H_0?

(i) $a \cdot p$ with $p = -\mathrm{i}\nabla$ and $\mathcal{D}(a \cdot p) = H^1(\mathbb{R}^3)$.

(ii) $a \cdot x$ with $\mathcal{D}(a \cdot x) = \{\psi \in \mathcal{H} \,|\, a \cdot x\, \psi \in \mathcal{H}\}$

8. Demonstrate that for all $\psi \in C_0^\infty(\mathbb{R}^3)$ the following inequalty holds:

$$\frac{1}{4} \int_{\mathbb{R}^3} |x|^{-2} \, |\psi(x)|^2 d^3x \leq \int_{\mathbb{R}^3} |\nabla \psi(x)|^2 d^3x. \tag{9}$$

9. Let the operator H in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3)$ be defined by

$$(H\psi)(x) = \big(|x|^2 + \frac{1}{4}\big)\psi(x) - \frac{1}{2\pi^2} \int_{\mathbb{R}^3} \frac{1}{|x-y|^2} \psi(y) d^3y \tag{10}$$

on the domain $\mathcal{D}(H) = \{\psi \in \mathcal{H} \,|\, \psi(x) = (|x|^2 + 1/4)^{-2} \phi(x)$ with $\phi \in C_0^\infty(\mathbb{R}^3)\}$.

(i) Show that by H a symmetric operator in $\mathcal{H}$ is defined.

(ii) Prove that H is bounded below. Hint: Consider the quadratic form $\langle \psi, \, H\psi \rangle$. You may use the integrals:

$$\int_0^\infty dt \, \frac{t \log\big((a \pm t)^2\big)}{(b^2 + t^2)^2} = \frac{\pm \pi a b + 2a^2 \log(a) + 2b^2 \log(b)}{2(a^2 + b^2) b^2} \tag{11}$$

(iii) Determine the (unnormalized) ground state wave function of H (Hint: In part (ii), can you make $\langle \psi, \, H\psi \rangle$ to attain its lower bound?) What can be concluded about the degeneracy of the ground state and the nodes of its wave function?