Mathematical Quantum Mechanics, 2014/15 Homework Problems, LMU

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3. As candidates for a momentum operator in the Hilbert space $\mathcal{H} = L^2(0,1)$, we introduce the operators $p_c = -\mathrm{i}d/dx$ (with c standing for min, 0, max) and their respective domains $\mathcal{D}(p_{\min}) = C_0^\infty(0,1)$, $\mathcal{D}(p_0) = \{\psi \in \mathcal{H} \,|\, \psi \in \mathrm{AC}(0,1), \psi' \in \mathcal{H}, \psi(0) = 0 = \psi(1)\}$, $\mathcal{D}(p_{\max}) = \{\psi \in \mathcal{H} \,|\, \psi \in \mathrm{AC}(0,1), \psi' \in \mathcal{H}\}$ (where the latter domain is just the Sobolev space $H^1_{(0,1)}$ on the interval $(0,1)$).

(i) Which of these operators are symmetric?

(ii) Show that $p^\star_{\min} = p_{\max}$ and $p^\star_{\max} = p_0$.

(iii) Prove that the closure of $p_{\min}$ is given by $\overline{p_{\min}} = p_0$ (Hint: you may use a relation between a closable densely defined linear operators A and its double adjoint $A^{\star\star}$).

(iv) Is any of the p_c an observable?

4. As an alternative to the Hamiltonian H considered in problem 2, in the Hilbert space $\mathcal{H} = L^2(0,1)$ we define $H_{\min}$ again by $H_{\min} = -d^2/dx^2$ but with $\mathcal{D}(H_{\min}) = C_0^\infty(0,1)$.

(i) Show that also $H_{\min}$ is a symmetric and nonnegative operator.

(ii) Demonstrate that the adjoint of $H_{\min}$ is given by $H_{\max} = -d^2/dx^2$ with $\mathcal{D}(H_{\max}) = H^2(0,1)$ (where you can use the following form of the second Sobolev space $H^2_{(0,1)} = \{\psi \in \mathcal{H} \,|\, \psi \in C^1(0,1), \psi' \in \mathrm{AC}(0,1), \psi'' \in \mathcal{H}\}$).

(iii) Construct the Friedrichs extension H_{F} of $H_{\min}$ and compare it to the operator H studied in problem 2. (Hint: The domain of H_{F} is given by $\mathcal{D}(H_{\mathrm{F}}) = \overline{\mathcal{Q}}(H_{\min}) \cap \mathcal{D}(H^{\star}_{\min})$; also observe that in this case $\overline{\mathcal{Q}}(H_{\min}) =$ closure of $C_0^{\infty}(0,1)$ in $\mathcal{H}$ with respect to the form norm $||\psi||_q = \sqrt{q[\psi] + ||\psi||^2}$ is identical to the closure of $C_0^{\infty}(0,1)$ with respect to the graph norm of the operator $p_{\min}$ from problem 3, and recall a result from problem 3).

5. Let $\psi \in \mathcal{S}(\mathbb{R})$ be given by $\psi(x) = \exp(-x^2/2)$, where $\mathcal{S}(\mathbb{R})$ denotes the Schwartz space over R.

(i) Compute the Fourier transform $\hat{\psi} = \mathcal{F}\psi$ of ψ. (Hint: If you use complex integration, rely on Cauchy's theorem.)

(ii) Employ scaling arguments to obtain the Fourier transform $\hat{\psi}_a$ of $\psi_a(x) = \exp(-ax^2/2)$ for $a > 0$.

6. Consider the function $V : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^+$ given by $V(x) = |x|^{-1}$ (viz., the Coulomb potential).

(i) Does V belong to a $L^p(\mathbb{R}^3)$ space for some $1 \leq p \leq \infty$?

(ii) Invoke scaling arguments to argue that the Fourier transform $\hat{V}$ of V (if it exists) must be of the form $\hat{V}(p) = \text{constant}\, |p|^{-2}$.

(iii) Show that for all $a > 0$ and $x \in \mathbb{R}^3 \setminus \{0\}$

$$\frac{\exp(-a|x|)}{|x|} = \frac{1}{2\pi^2} \int_{\mathbb{R}^3} \frac{\exp(-\mathrm{i}p \cdot x)}{|p|^2 + a^2}\, d^3p \tag{5}$$

(iv) Deduce that (iii) leads to

$$\int_{\mathbb{R}^3} \frac{\psi(x)}{|x|}\, d^3x = \sqrt{\frac{2}{\pi}} \int_{\mathbb{R}^3} \frac{\hat{\psi}(p)}{|p|^2} d^3p \tag{6}$$

for all $\psi \in \mathcal{S}(\mathbb{R})$.