Mathematical Quantum Mechanics, 2014/15
Homework Problems, LMU

Issued: January 7, 2015; deadline for handing in the solutions:
Wednesday, January 14, 2015, 4 pm

32. For a neutral atom and the corresponding Thomas-Fermi minimizer $\rho_{\rm at}$ set $T = (3/5)\gamma_{\rm TF} \int \rho_{\rm at}^{5/3}$, and $V_{\rm ne} = Z \int \rho_{\rm at}/|x|$, and $V_{\rm ee} = D(\rho_{\rm at}, \rho_{\rm at})$. Show that the following relations hold.

$$\begin{aligned} 0 &= 2T - V_{\rm ne} + V_{\rm ee}, \\ 0 &= (5/3)T - V_{\rm ne} + 2V_{\rm ee}. \end{aligned} \qquad (37)$$

Compute the ensuing ratio $(-V_{\rm ne})/T/(-E_{\rm TF})/V_{\rm ee}$ with $E_{\rm TF} = T - V_{\rm ne} + V_{\rm ee}$.

33. Consider the N-body bosonic system described by the Hamiltonian

$$H = \sum_{j=1}^{N} -\Delta_j + \sum_{1 \le i < j \le N} e_i e_j V(x_i - x_j) \qquad (38)$$

for the Coulomb potential $V(x) = |x|^{-1}$ and the charges $e_j \in \mathbb{R}$, $j = 1, \ldots N$. Show that if V is regularized at $x = 0$, e.g., by replacing V by $W(x) = \big(1 - \exp(-m|x|)\big)|x|^{-1}$ with $m > 0$, then H can be bounded below by

$$H \ge \sum_{j=1}^{N} -\Delta_j - \frac{N}{2} e_0^2 W(0) \ge \text{const } N \qquad (39)$$

where $e_0 = \max_{1 \le j \le N} |e_j|$.

34. Prove that the annihilation and creation operators $a, a^\star$, respectively, defined in the lecture are mutually adjoint, and that the second quantized

Hamiltonian $H = \sum_{i,j} \langle e_i, (-\Delta + v) e_j \rangle a_i^\star a_j + (1/2) \sum_{i,j,k,l} \langle e_i \otimes e_j, w e_k \otimes e_l \rangle a_i^\star a_j^\star a_l a_k$ is symmetric and leaves the N-particle spaces invariant and acts in the N-particle spaces as $\sum_{i=1}^{N} (-\Delta_i - Z|x_i|^{-1}) + \sum_{1 \leq i < j \leq N} |x_i - x_j|^{-1}$. Here v stands for the one-body potential $(v\psi)(x) = -Z|x|^{-1}\psi(x)$ and w for the two-body interaction $(w\psi)(x,y) = |x-y|^{-1}\psi(x,y)$.