Mathematical Quantum Mechanics, 2014/15
Homework Problems, LMU

Issued: October 7, 2014; deadline for handing in the solutions:
October 14, 2014, 4 pm

1. Let $\mathrm{Mat}_{n,n}$ stand for the set of n, n matrices with complex elements.

(i) Prove or disprove that if $1 \leq n < \infty$, then there exist $A, B \in \mathrm{Mat}_{n,n}$ and $c \in \mathbb{C} \setminus \{0\}$ such that

$$[A, B] := AB - BA = c, \tag{1}$$

where on the right hand side the unit matrix is not displayed explicitly. Could you apply the same arguments of your proof if $n = \infty$?

(ii) Show that $[A, B] = \mathrm{i}$ would imply that $A^k B - BA^k = \mathrm{i}kA^{k-1}$ for all strictly positive integers $k \in \mathbb{N}$.

(iii) Let $\ell_2 = \{(x_i)_{i=1}^{\infty} \,|\, x_i \in \mathbb{C},\ \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$ be the Hilbert space of square-summable sequences $(x_i)_{i=1}^{\infty} = (x_1, x_2, \ldots)$. Linear operators A on ℓ_2 may be represented as infinite matrices such that $A(x_i)_{i=1}^{\infty} = (\sum_{j=1}^{\infty} A_{ij} x_j)_{i=1}^{\infty}$. Prove that by

$$||A|| = \sup_{||(x_i)_{i=1}^{\infty}||=1} ||A(x_i)_{i=1}^{\infty}|| \tag{2}$$

a norm on the set of bounded linear operators on ℓ_2 is defined, and that $||AB|| \leq ||A||\,||B||$ holds.

(iv) Employ (ii) and the last inequality to demonstrate that $[A, B] = \mathrm{i}$ for linear operators A, B on ℓ_2 implies $k \leq 2||A||\,||B||$ for all $k \in \mathbb{N}$ and conclude that the relation $[A, B] = \mathrm{i}$ cannot be achieved by bounded operators on ℓ_2.

2. The example of a particle in a one-dimensional infinite square well is studied in every standard quantum text book, unfortunately not always in a mathematically correct way. Mathematically, the infinite potential walls at $x = 0$ and $x = 1$ are incorporated by imposing Dirichlet boundary conditions, i.e., by considering the Hilbert space $\mathcal{H} = L^2(0,1)$ and the Hamiltonian

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} \tag{3}$$

on the domain $\mathcal{D}(H) = \{\psi \in \mathcal{H} \mid \psi \in C^1(0,1), \psi' \in \mathrm{AC}(0,1), \psi'' \in L^2(0,1), \psi(0) = 0 = \psi(1)\}$, where $\mathrm{AC}(0,1)$ are the absolutely continuous functions on $(0,1)$. For convenience, we may assume that $\hbar = 2m = 1$.

(i) Prove that H is a nonnegative operator, i.e., $\langle \psi,\, H\psi \rangle \geq 0$ for all $\psi \in \mathcal{D}(H)$.

(ii) As a rigorous alternative of the (frequently but not convincingly) invoked argument of the uncertainty relation to infer the existence of a nonvanishing zero-point energy for H, show (without solving explicitly the Schrödinger equation) that the energy of the system is bounded below by a positive constant, viz., for all $\psi \in \mathcal{D}(H)$, $\psi \neq 0$, and assuming $\hbar = 2m = 1$, we have

$$\frac{\langle \psi,\, H\psi \rangle}{\langle \psi, \psi \rangle} \geq 1. \tag{4}$$

(iii) Compare the lower bound obtained in (ii) with the optimal lower bound for H, that is, with the ground state energy of H (again for $\hbar = 2m = 1$).