or upon introducing the three-dimensional vectors

$$A = (1, y, z) \quad \text{and} \quad B = (1, y + v, z + w),$$

that

$$|B| \geq |A| + \frac{(yv + zw)}{|A|} = \frac{|A|^2 + (yv + zw)}{|A|}$$
$$= \frac{1 + y(y + v) + z(z + w)}{|A|} = \frac{A \cdot B}{|A|},$$

where the dot denotes the scalar product of the vectors.

Since $A \cdot B = |A||B| \cos(A, B) \leq |A||B|$ with equality *iff* A and B are codirected, it is seen that (11) does hold with equality *iff* $(1, y, z)$ and $(1, y + v, z + w)$ are codirected, i.e., *iff* $v = w = 0$. Thus, $f(y, z) = \sqrt{1 + y^2 + z^2}$ is strongly convex on $\mathbb{R}^2$.

EXAMPLE 11. If $0 < p \in C[a, b]$, then

$$f(\underline{x}, y, z) = p(\underline{x})\sqrt{1 + y^2 + z^2}$$

is strongly convex on $[a, b] \times \mathbb{R}^2$ (Fact 2 and Example 10.)

EXAMPLE 12. When $b \neq 0$, then $f(y, z) = \sqrt{y^2 + b^2z^2}$ has derivatives

$$f_y(y, z) = \frac{y}{\sqrt{y^2 + b^2z^2}} \quad \text{and} \quad f_z(y, z) = \frac{b^2z}{\sqrt{y^2 + b^2z^2}},$$

which are discontinuous at the origin. However, on the restricted set $\mathbb{R}^2 \sim \{(0, 0)\}$ this function is again convex but not strongly convex. (See Problem 3.24.)

(Problems 3.1–3.19)

§3.4. Applications

In this section we show that convexity is present in problems from several diverse fields—at least after suitable formulation—and use previous results to characterize their solutions. Applications, presented in order of increasing difficulty—and/or sophistication, are given which characterize geodesics on a cylinder, a version of the brachistochrone, Newton's profile of minimum drag, an optimal plan of production, and a form of the minimal surface. Other applications in which convexity can be used with profit will be found in Problems 3.20 et seq.

(a) Geodesics on a Cylinder

To find the geodesics on the surface of a right circular cylinder of radius 1 unit, we employ, naturally enough, the cylindrical coordinates (θ, z) shown in Figure 3.1 to denote a typical point. It is obvious that the geodesic joining points $P_1 = (\theta_1, z_1)$, $P_2 = (\theta_1, z_2)$ is simply the vertical segment connecting them. Thus it remains to consider the case where $P_2 = (\theta_2, z_2)$ with $\theta_2 \neq \theta_1$; a little thought shows that by relabelling if necessary, we can suppose that $0 < \theta_2 - \theta_1 \leq \pi$, and consider those curves which admit representation as the graph of a function $z \in \mathscr{D} = \{z \in C^1[\theta_1, \theta_2]: z(\theta_j) = z_j, j = 1, 2\}$.

The spatial coordinates of such a curve are

$$(x(\theta), y(\theta), z(\theta)) = (\cos\theta, \sin\theta, z(\theta)),$$

so that when $z \in \mathscr{D}$, the resulting curve has the length

$$L(z) = \int_{\theta_1}^{\theta_2} \sqrt{x'(\theta)^2 + y'(\theta)^2 + z'(\theta)^2}\, d\theta = \int_{\theta_1}^{\theta_2} \sqrt{1 + z'(\theta)^2}\, d\theta.$$

With an obvious change in variables, this integrand corresponds to the function of §3.3, Example 3, which is strongly convex. Thus by Corollary 3.8, we conclude that

(3.12) *Among curves which admit representation as the graph of a function $z \in \mathscr{D}$, the minimum length is given uniquely for that represented by the function*

$$z_0(\theta) = z_1 + m(\theta - \theta_1) \quad \textit{for } m = \frac{z_1 - z_2}{\theta_1 - \theta_2},$$

which describes the circular helix joining the points.

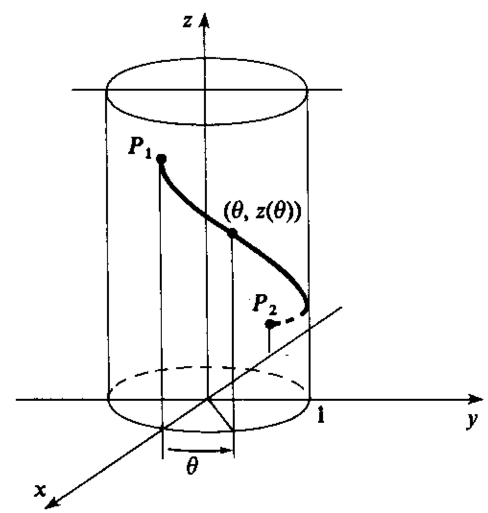

Figure 3.1

(If the cylinder were "unrolled," this would correspond to the straight line joining the points.) Plants take helical paths when climbing around cylindrical supporting stakes toward the sun [Li].

(b) A Brachistochrone

For our next application, we return to the brachistochrone of §1.2(a). As formulated there, the function $T(y)$ is not of the form covered by Theorem 3.5. (Why not?) However, if we interchange the roles of x and y and consider those curves which admit representation as the graph of a function $y \in \mathscr{D} = \{y \in C^1[0, x_1]: y(0) = 0, y(x_1) = y_1\}$ (with x_1 and y_1 both positive) as in Figure 3.2, then in the *new* coordinates, the same analysis as before gives for each such curve the transit time

$$T(y) = \int_0^{x_1} \sqrt{\frac{1 + y'(x)^2}{2gx}}\, dx,$$

which has the strongly convex integrand function of §3.3, Example 4, with $r = 1$ and $p(x) = (2gx)^{-1/2}$ on $(0, x_1]$. Now $p(x)$ is positive and integrable on $[0, x_1]$ and although it is not continuous (at 0), Theorem 3.7 remains valid. (See Problem 3.21.)

Thus we know that among such curves, the minimum transit time would be given uniquely by each $y \in \mathscr{D}$ which makes

$$\frac{y'(x)}{\sqrt{x}\sqrt{1 + y'(x)^2}} \equiv \frac{1}{c} \quad \text{for } \textit{some} \text{ constant } c.$$

Squaring both sides gives the equation

$$\frac{y'(x)^2}{1 + y'(x)^2} = \frac{x}{c^2}$$

or

$$y'(x)^2 = \frac{x}{c^2 - x}. \tag{12}$$

Thus $y'(0) = 0$.

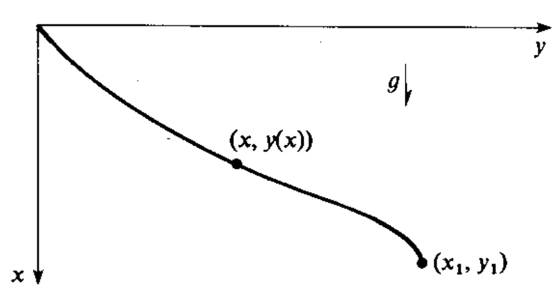

Figure 3.2

If we introduce the new independent variable θ through the relation $x(\theta) = (c^2/2)(1 - \cos\theta) = c^2 \sin^2(\theta/2)$, then $\theta = 0$ when $x = 0$, and for $\theta < \pi$, θ increases with x. By the chain rule

$$\frac{dy}{d\theta} = y'(x)x'(\theta) = y'(x)\left(\frac{c^2}{2}\sin\theta\right),$$

and equation (12) becomes

$$\left(\frac{dy}{d\theta}\right)^2 \frac{4}{(c^2 \sin\theta)^2} = y'(x)^2 = \frac{1 - \cos\theta}{1 + \cos\theta},$$

or

$$\frac{dy}{d\theta} = \frac{c^2}{2}\sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}}\sin\theta = \frac{c^2}{2}(1 - \cos\theta).$$

Hence $y(\theta) = (c^2/2)(\theta - \sin\theta) + c_1$, and the requirement $y(0) = 0$ shows that $c_1 = 0$.

Upon replacing the unspecified constant c by $\sqrt{2}\,c$, we see that the minimum transit time would be given parametrically by a curve of the form

$$\begin{cases} x(\theta) = c^2(1 - \cos\theta), \\ y(\theta) = c^2(\theta - \sin\theta), \end{cases} \quad 0 \le \theta \le \theta_1, \tag{13}$$

provided that c^2 and θ_1 can be found to make $x(\theta_1) = x_1$, $y(\theta_1) = y_1$. The curve described by these equations is the cycloid with cusp at (0, 0) which would be traced by a point on the circumference of a disk of radius c^2 as it rolls along the y axis from "below" as shown in Figure 3.3.

For $\theta > 0$, the ratio $y(\theta)/x(\theta) = (\theta - \sin\theta)/(1 - \cos\theta)$ has the limiting value $+\infty$ as $\theta \uparrow 2\pi$, and by L'Hôpital's rule it has the limiting value of 0 as $\theta \downarrow 0$. Its derivative is

$$\frac{(1 - \cos\theta)^2 - \sin\theta(\theta - \sin\theta)}{(1 - \cos\theta)^2} = \frac{2(1 - \cos\theta) - \theta\sin\theta}{(1 - \cos\theta)^2},$$

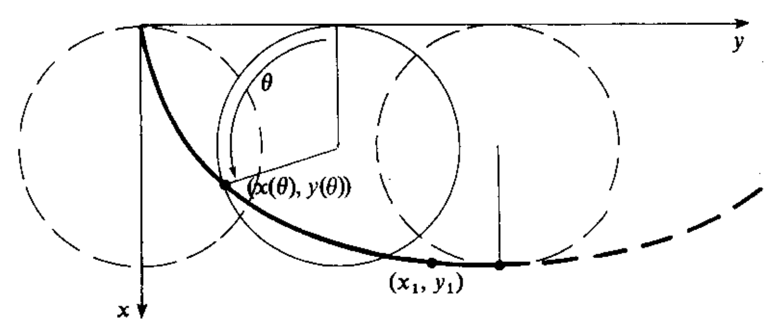

Figure 3.3

which utilizing the half angle formulae may be rewritten as

$$\frac{\cos\theta/2}{\sin^3\theta/2}\left(\tan\frac{\theta}{2}-\frac{\theta}{2}\right)\qquad(\theta\neq\pi),$$

and thus is positive for $0<\theta<2\pi$. (Why?) $y(\theta)/x(\theta)$ is positive, increases strictly from 0 to $+\infty$ as θ increases from 0 to 2π, and hence from continuity (through the intermediate value theorem of §A.1), assumes each positive value precisely once. In particular, there is a unique $\theta_1\in(0,2\pi)$ for which $y(\theta_1)/x(\theta_1)=y_1/x_1$, and for this θ_1, choosing $c^2=x_1/(1-\cos\theta_1)$ will guarantee the existence of a (unique) cycloid joining (0, 0) to (x_1, y_1).

Unfortunately, as Figure 3.3 shows, the associated curve can be represented in the form $y=y(x)$ only when $\theta_1\leq\pi$, i.e., when $y_1/x_1\leq\pi/2$. Moreover, the associated function $y\in C^1[0,x_1]$ only when $y_1/x_1<\pi/2$, since the tangent line to the cycloid must be horizontal at the lowest point on the arch. Nevertheless, we do have a nontrivial result:

(3.13) *When $y_1/x_1<\pi/2$, among all curves representable as the graph of a function $y\in C^1[0,x_1]$ which join (0, 0) to (x_1, y_1), the cycloid provides uniquely the least time of descent.*

Thus we confirm Galileo's belief that the brachistochrone is *not* the straight line and support the classical assertion by Newton and the Bernoullis that it must always be a cycloid.

It is not too difficult to extend our analysis to the case $y_1/x_1=\pi/2$ (see Problem 3.22*), and it may seem physically implausible to consider curves which fall below their final point or those which have horizontal sections (i.e., those which *cannot* be expressed in the form $y=y(x)$) as candidates for the brachistochrone. However, it is true that the brachistochrone is always the cycloid, but a proof for the general case must be deferred until we have the far more sophisticated tools of Chapter 9.

(c) A Profile of Minimum Drag

One of the first problems to be attacked by a variational approach was that propounded by Newton in his *Principia* (1686) of finding the profile of [the shoulder of] a *projectile* of *revolution* which would offer minimum resistance (or drag) when moved in the direction of its axis at a constant (unit) speed in water.

We adopt the coordinates and geometry shown in Figure 3.4, and postulate with Newton that the resisting pressure at a surface point on the shoulder is proportional to the *square* of the *normal* component of its velocity. Then, if ψ denotes the angle between the positive x axis and the *tangent* to a point on a

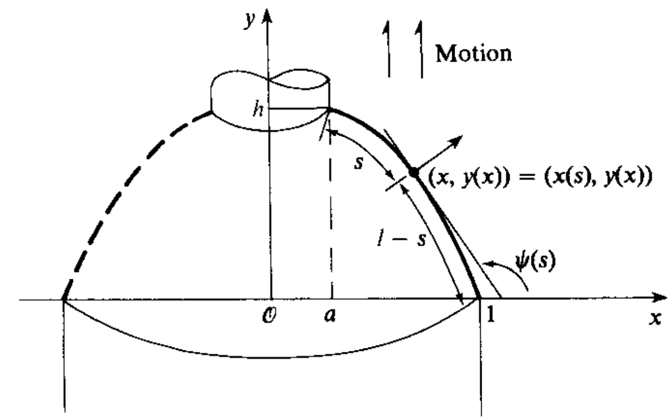

Figure 3.4

meridional curve of length l whose rotation determines the surface of the shoulder, we wish to minimize

$$\int_0^l \cos^2(\pi-\psi(s))2\pi x(s)\cos\psi(s)\,ds.$$

Since $\cos\psi(s)=x'(s)$ while $1+y'(x)^2=\sec^2\psi(s)$, we evidently wish to minimize

$$F(y)=\int_a^1 x(1+y'(x)^2)^{-1}\,dx,$$

on

$$\mathcal{D}=\{y\in C^1[a,1]: y(a)=h,\ y(1)=0,\ y(x)\geq 0\},$$

where we suppose that the *positive* constants $a<1$ and h are given; ($a=0$ is excluded for reasons which will emerge). Now, if

$$f(x,z)=\frac{x}{1+z^2},\quad\text{then } f_z(x,z)=\frac{-2zx}{(1+z^2)^2},$$

and for $x>0$:

$$f_{zz}(x,z)=\frac{2x(3z^2-1)}{(1+z^2)^3}>0,\quad\text{when } |z|>\frac{1}{\sqrt{3}}.$$

On physical grounds we expect $y'\leq 0$ (Why?), and, by Proposition 3.10, $f(\underline{x},z)$ is strongly convex on $[a,1]\times(-\infty,-1/\sqrt{3}]$. Hence from Theorem 3.7, we know that if

$$y_0\in\mathcal{D}'=\{y\in\mathcal{D}: y'(x)\leq-1/\sqrt{3},\ x\in[a,1]\}$$

makes

$$f_z(x,y'(x))=\frac{-2xy'(x)}{(1+y'(x))^2}=\text{const.}=\frac{\sqrt{2}}{c},$$

say, for a *positive* constant c, then it minimizes F on $\mathscr{D}'$ uniquely. Upon squaring and rewriting, we wish for $u = 1 + y'(x)^2$ that $u^2 - 2c^2x^2u + 2c^2x^2 = 0$. Solving for u, we obtain $u = c^2x^2 \pm cx(c^2x^2 - 2)^{1/2}$, provided that $c^2a^2 \geq 2$ (and *this* excludes $a = 0$). Then $-\sqrt{2}cxy'(x) = u^2 = 2c^2x^2(u - 1)$ so that

$$y'(x) = -\sqrt{2}cx[(c^2x^2 - 1) \mp cx(c^2x^2 - 2)^{1/2}],$$

and upon integration, incorporating the *outer* boundary condition $y(1) = 0$, we get

$$y(x) = 2^{-3/2}c\left\{c^2(1 - x^4) - 2(1 - x^2) \mp 4c\int_x^1 t^2(c^2t^2 - 2)^{1/2}\,dt\right\}, \quad (14)$$

where c is to be determined if possible to satisfy the inner boundary condition $y(a) = h$, while keeping $y'(x) \leq -1/\sqrt{3}$. Although the remaining integration can also be performed, we shall not pursue this approach further except to remark that in view of the uniqueness, at most one solution $y = y_0 \in \mathscr{D}'$ is possible. Hence, depending on the particular geometrical constants, it suffices to consider only one of the signs in (14). In practice, it would be easier to choose $c > 2$, and then determine the values of a and $h = y(a)$ which can be attained by numerical integration in (14), while keeping $y'(x) \leqq -1/\sqrt{3}$. Each nontrivial solution thus obtained provides the profile of minimum drag *at least among those in* $\mathscr{D}'$, and this could be used in designing a torpedo or some other missile moving in a medium *for which Newton's resistance law is a reasonable assumption.*[1] Alternative drag postulates can be investigated by the same approach. See [P] for a more thorough discussion.

In the preceding sections, we have explored a rather complete theory for analyzing the classical problem of characterizing the minimum of the function

$$F(y) = \int_a^b f(x, y(x), y'(x))\,dx,$$

where $f(\underline{x}, y, z)$ is suitably defined and convex, over all functions $y \in C^1[a, b]$ with prescribed end point values.

We examine now two problems which are not of the same form but which admit solution through the same considerations.

(d) An Economics Problem

All of the classical optimization problems arose in the development of physics and are usually concerned with optimizing one of the fundamental quantities, length, time, or energy under various conditions. For a change of interest, we

[1] Newton himself believed that his results might be applicable in the design of a ship's hull. See [Fu].

will consider a more contemporary problem of production planning (whose statement is taken from [Sm]).

From previous data a manufacturing company with "continuous" inventory has decided that with an initial inventory $\mathscr{I}(0) = \mathscr{I}_0$ and a projected sales rate $S = S(t)$ over a time interval $[0, T]$, the best production rate at time t is given by a function $\mathscr{P} = \mathscr{P}(t)$. Assuming that loss during storage occurs at a rate which is a *fixed* proportion, α, of the associated inventory $\mathscr{I}(t)$ at time t (perhaps through spoilage), and the rest is sold at the projected rate $S(t)$, then we should have at time t, the simple differential relation

$$\mathscr{I}'(t) = \mathscr{P}(t) - S(t) - \alpha\mathscr{I}(t)$$
$$(\text{or} \quad \mathscr{P}(t) = \mathscr{I}'(t) + \alpha\mathscr{I}(t) + S(t)).$$

Now suppose that it wishes to maintain the same sales rate $S(t)$ over a period $[0, T]$ but its actual initial inventory $I(0) = I_0 \neq \mathscr{I}_0$. Then from the assumed continuity, each projected production rate function $P = P(t)$ results in an inventory $I(t)$ at time t which differs from $\mathscr{I}(t)$ (at least in a neighborhood of 0). With the same percentage loss, we would have as above,

$$P(t) = I'(t) + \alpha I(t) + S(t).$$

As a consequence, the company will experience additional operating costs (perhaps due to handling and storage problems); these costs might be estimated by a function such as

$$C = \int_0^T [\beta^2(I - \mathscr{I})^2(t) + (P - \mathscr{P})^2(t)]\,dt,$$

which takes into account the deviations in both inventory I and associated production rate P from their "ideal" counterparts. (β is a constant which adjusts proportions.) This is rather a crude measure of cost, but it possesses analytical advantages. Moreover, since both $\mathscr{I}$ and $\mathscr{P}$ are known, while P is determined by I as above, the cost function C may be regarded as

$$\begin{aligned} C(I) &= \int_0^T [\beta^2(I(t) - \mathscr{I}(t))^2 + (I'(t) - \mathscr{I}'(t) + \alpha(I(t) - \mathscr{I}(t)))^2]\,dt \\ &= \int_0^T [(\alpha^2 + \beta^2)(I - \mathscr{I})^2(t) + (I' - \mathscr{I}')^2(t)]\,dt \\ &\quad + 2\alpha\int_0^T (I - \mathscr{I})(I - \mathscr{I})'(t)\,dt, \end{aligned} \quad (15)$$

which should be minimized over all functions $I \in C^1[0, T]$ for which $I(0) = I_0$. Integrating the last term and introducing

$$\gamma^2 = \alpha^2 + \beta^2, \quad (16)$$

there results

$$C(I) + \alpha(I_0 - \mathscr{I}_0)^2 = \int_0^T [\gamma^2(I - \mathscr{I})^2(t) + (I' - \mathscr{I}')^2(t)]\,dt + \alpha(I - \mathscr{I})^2(T).$$

It is now natural to introduce the inventory deviation function $y(t) = I(t) - \mathscr{I}(t)$ which has the *known* initial value $y(0) = I_0 - \mathscr{I}_0 = y_1$, say, and consider instead the equivalent problem of minimizing the modified cost function

$$\tilde{C}(y) = \int_0^T [\gamma^2 y(t)^2 + y'(t)^2]\,dt + \alpha y^2(T) \tag{17}$$

over $\mathscr{D} = \{y \in C^1[0, T]: y(0) = y_1 \text{ prescribed}\}$.

The presence of the term $\alpha y^2(T)$ prevents this function from being of the classical form studied in §3.2. However, the general approach employed there is suggested, since the integrand function, viz.,

$$f(y, z) = \gamma^2 y^2 + z^2$$

is clearly strongly convex on $\mathbb{R}^2$, so that $\tilde{C}$ is strictly convex for $\alpha > 0$. (See Proposition 3.2 and Example 4 of §3.1.)

Now, $\forall\, y, v \in C^1[0, T]$,

$$\delta\tilde{C}(y; v) = 2\int_0^T [\gamma^2 y(t)v(t) + y'(t)v'(t)]\,dt + 2\alpha y(T)v(T),$$

and a minimizing function would be given by a $y \in \mathscr{D}$ for which

$$\delta\tilde{C}(y; v) = 0, \qquad \forall\, y + v \in \mathscr{D}.$$

Introducing $\mathscr{D}_0 = \{v \in C^1[0, T]: v(0) = 0\}$, naive inspection suggests consideration of a $y \in \mathscr{D}$ which makes the last integral vanish $\forall\, v \in \mathscr{D}_0$ and for which $y(T) = 0$. However, it is not evident how to force the vanishing of the integral. Instead we try to find a $y \in \mathscr{D} \cap C^2[0, T]$, and integrate the *second* term by parts to get the equation

$$\delta\tilde{C}(y; v) = 2\int_0^T [\gamma^2 y(t) - y''(t)]v(t)\,dt + 2[y'(T) + \alpha y(T)]v(T),$$

(where we have incorporated the vanishing of v at 0).

Now it *is* clear that $\delta\tilde{C}(y; v) = 0, \forall\, v \in \mathscr{D}_0$, provided that y satisfies the differential equation

$$y'' - \gamma^2 y = 0 \tag{18}$$

and the natural boundary condition

$$y'(T) + \alpha y(T) = 0. \tag{19}$$

(In more familiar terms, (19) simply requires that the terminal production rate, $P(T) = \mathscr{P}(T)$. Indeed, in general, $P - \mathscr{P} = y' + \alpha y$.) (Why?) In addition to (19), y must also satisfy the given boundary condition $y(0) = y_1$.

The general solution of the differential equation (18) is well known to be given by

$$y_0(t) = c_1 e^{\gamma t} + c_2 e^{-\gamma t}, \quad \text{with} \quad y_0'(t) = \gamma(c_1 e^{\gamma t} - c_2 e^{-\gamma t}), \tag{20}$$

but as usual in these problems, it is not guaranteed that the constants c_1 and c_2 can be found so that $y(t)$ satisfies the boundary conditions. We require that

$$y_0(0) = y_1 = c_1 + c_2,$$

and

$$\begin{aligned} 0 &= y_0'(T) + \alpha y_0(T) \\ &= c_1(\gamma + \alpha)e^{\gamma T} + c_2(-\gamma + \alpha)e^{-\gamma T}, \end{aligned}$$

or that

$$0 = c_1(\gamma + \alpha)e^{2\gamma T} - c_2(\gamma - \alpha).$$

From this last equation, the ratio

$$\rho \overset{\text{def}}{=} \frac{\gamma + \alpha}{\gamma - \alpha} e^{2\gamma T} = \frac{c_2}{c_1} \tag{21}$$

is specified, and for this ρ the choices $c_1 = y_1/(1 + \rho)$, $c_2 = y_1\rho/(1 + \rho)$ will satisfy both conditions. This gives the desired conclusion:

(3.14) *Among all inventory functions $I \in C^1[0, T]$ with α, β, and $I(0) = I_0$ prescribed, that given by*

$$I(t) = \mathscr{I}(t) + \frac{(I_0 - \mathscr{I}_0)}{1 + \rho}(e^{\gamma t} + \rho e^{-\gamma t}), \tag{22}$$

with ρ, γ determined by (21) *and* (16), *respectively, will provide uniquely the minimum cost of operation as assessed by* (15).

Moreover, in this case the minimum cost can easily be computed. Indeed from (17)

$$\tilde{C}(y) = \int_0^T [\gamma^2 y(t)^2 + y'(t)^2]\,dt + \alpha y^2(T)$$

and when $y \in C^2[0, T]$, integration of the *second* term by parts gives

$$\tilde{C}(y) = \int_0^T (\gamma^2 y - y'')(t)y(t)\,dt + [y'(T) + \alpha y(T)]y(T) - y'(0)y(0). \tag{23}$$

Since $y(t) = I(t) - \mathscr{I}(t)$, the minimizing y is seen from (22) to be

$$y_0(t) = \frac{(I_0 - \mathscr{I}_0)}{1 + \rho}(e^{\gamma t} + \rho e^{-\gamma t}). \tag{24}$$

Evaluating (23) for $y = y_0$ we see that the integral vanishes because y_0 satisfies the differential equation (18), and the next term also vanishes because y_0 fulfills the boundary condition (19). From the other boundary condition, viz., $y_0(0) = y_1$, we have

$$\tilde{C}_{\min} = \tilde{C}(y_0) = -y_1 y_0'(0).$$

Thus differentiating (24) and evaluating at 0:

$$\tilde{C}_{\min} = y_1 \gamma \frac{(I_0 - \mathscr{I}_0)}{1 + \rho}(\rho - 1),$$

or finally recalling that $y_1 = I_0 - \mathscr{I}_0$:

$$C_{\min} = \tilde{C}_{\min} - \alpha y_1^2 = y_1^2\left[\frac{\gamma(\rho - 1)}{\rho + 1} - \alpha\right]$$

$$= (I_0 - \mathscr{I}_0)^2\left[\frac{\gamma(\rho - 1)}{\rho + 1} - \alpha\right],$$

and this expression shows the effects of various choices of α, β, T, and $\mathscr{I}(0)$ on the minimum cost of operation. Observe that it is independent of the sign of the initial inventory deviation.

(e) Minimal Area Problem

Our final example extends the methods of this chapter to a problem in higher dimensions, namely, that of Plateau. In the simplified version formulated in §1.4(b), given a bounded domain $D \subset \mathbb{R}^2$, and a prescribed smooth boundary function γ, we seek a function $u \in C^1(\bar{D})$ which has these boundary values and minimizes the surface area function

$$S(u) = \iint_D \sqrt{1 + u_x^2 + u_y^2}\, dx\, dy.$$

Introducing $\mathscr{D} = \{u \in C^1(\bar{D}) \text{ with } u|_{\partial D} = \gamma\}$ and $\mathscr{D}_0 = \{v \in C^1(\bar{D}) \text{ with } v|_{\partial D} = 0\}$, we see that this is equivalent to finding a $u \in \mathscr{D}$ for which

$$S(u + v) - S(u) \geq 0, \qquad \forall\, v \in \mathscr{D}_0.$$

Now, the three-dimensional vector inequality used in establishing the strong convexity of $f(y, z) = \sqrt{1 + y^2 + z^2}$ (see Example 10 of §3.3), shows that at each point in D:

$$\sqrt{1 + (u_x + v_x)^2 + (u_y + v_y)^2} - \sqrt{1 + u_x^2 + u_y^2} \geq \frac{u_x v_x + u_y v_y}{\sqrt{1 + u_x^2 + u_y^2}},$$

with equality *iff* $v_x = v_y = 0$. Hence, from the assumed continuity:

$$S(u + v) - S(u) \geq \delta S(u; v) = \iint_D \frac{u_x v_x + u_y v_y}{\sqrt{1 + u_x^2 + u_y^2}}\, dx\, dy$$

(as in §2.4, Example 9) with equality *iff* $v \equiv 0$ (since $v_x = v_y \equiv 0$ in the *domain* $D \Rightarrow v = \text{const.} = v|_{\partial D} \equiv 0$). Thus S is strictly convex on $\mathscr{D}$, and again we would seek $u \in \mathscr{D}$ for which $\delta S(u; v)$ vanishes, $\forall\, v \in \mathscr{D}_0$. Such a u would provide

the unique minimizing function for S on $\mathscr{D}$. It would, of course, suffice if we could find a u which is even smoother; in particular, if we could find a $u \in \mathscr{D} \cap C^2(D)$ which has these properties.

For $u \in \mathscr{D} \cap C^2(D)$, both

$$U \stackrel{\text{def}}{=} \frac{u_x}{\sqrt{1 + u_x^2 + u_y^2}} \quad \text{and} \quad W \stackrel{\text{def}}{=} \frac{u_y}{\sqrt{1 + u_x^2 + u_y^2}},$$

are in $C^1(D)$ so that the integrand of $\delta S(u; v)$ may be rewritten as $Uv_x + Wv_y = (Uv)_x + (Wv)_y - (U_x + W_y)v$. Now, if we assume that Green's theorem holds for the domain D ([Fl]), then

$$\iint_D [(Uv)_x + (Wv)_y]\, dx\, dy = \int_{\partial D} [(Uv)\, dy - (Wv)\, dx],$$

and for $v \in \mathscr{D}_0$, the line integral vanishes. Thus for $v \in \mathscr{D}_0$,

$$\delta S(u; v) = -\iint_D (U_x + W_y)v\, dx\, dy, \tag{25}$$

and by Proposition 3.3 it is obvious that a minimum area would be given uniquely by each $u \in \mathscr{D} \cap C^2(D)$ which satisfies the *partial differential equation*

$$U_x + W_y = 0 \quad \text{in } D;$$

or upon substitution and simplification, which satisfies the second-order partial differential equation

$$(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0. \tag{26}$$

Equation (26) is called the *minimal surface equation* and it has been studied extensively. Our uniqueness argument shows that this equation cannot have more than one solution u in $\mathscr{D}$, but the existence of a solution depends upon a geometric condition on D:

(3.15) *A domain D is said to be convex when it contains the line segment joining each pair of its points.* A disk is convex while an annulus is not. If the domain D is not convex, it is known that (26) does not always have a solution in the required set $\mathscr{D} \cap C^2(D)$, and we can draw no additional conclusions from the analysis given here. However, it is also known that if D is convex, then (26) has a solution in $\mathscr{D} \cap C^2(D)$ for arbitrary *smooth* γ, which thus describes uniquely the minimal surface; i.e., the surface of minimal area spanning the contour described by the graph of the boundary function γ, among all C^1 surfaces ([Os]). (Actually it does so among all *piecewise* C^1 surfaces, those described by the graph of a piecewise C^1 function $\hat{u}$, which admit internal "roof-shaped" sections. With appropriate definitions, the methods of Chapter 7 can be extended to establish this fact.)

(Problems 3.20–3.26)

§3.5. Minimization with Convex Constraints

Convexity may also be of advantage in establishing the minima of functions J when constrained to the level sets of other functions G (as in the isoperimetric problem). In the formulation suggested by Proposition 2.3, the next result is apparent.

(3.16) **Theorem.** *If D is a domain in $\mathbb{R}^2$, such that for some constants λ_j, $j = 1, 2, \ldots, N$, $f(\underline{x}, y, z)$ and $\lambda_j g_j(\underline{x}, y, z)$ are convex on $[a, b] \times D$* [and at least *one* of these functions is strongly convex on this set], *let*

$$\tilde{f} = f + \sum_{j=1}^{N} \lambda_j g_j.$$

Then each solution y_0 of the differential equation

$$\frac{d}{dx} \tilde{f}_z[y(x)] = \tilde{f}_y[y(x)] \quad \text{on } (a, b)$$

minimizes

$$F(y) = \int_a^b f[y(x)]\, dx$$

[uniquely] *on*

$$\mathcal{D} = \{y \in C^1[a, b]: y(a) = y_0(a),\, y(b) = y_0(b);\, (y(x), y'(x)) \in D\}$$

under the constraining relations

$$G_j(y) \stackrel{\text{def}}{=} \int_a^b g_j[y(x)]\, dx = G_j(y_0), \qquad j = 1, 2, \ldots, N.$$

PROOF. By construction (and 3.11(1)) $\tilde{f}(\underline{x}, y, z)$ is [strongly] convex on $[a, b] \times D$, so that by Theorem 3.5, y_0 minimizes

$$\tilde{F}(y) = \int_a^b \tilde{f}[y(x)]\, dx = F(y) + \sum_{j=1}^{N} \lambda_j G_j(y)$$

[uniquely] on $\mathcal{D}$. Now apply Proposition 2.3. □

(3.17) **Remark.** Theorem 3.16 offers a valid approach to minimization in the presence of given isoperimetric constraints as we shall show by example. However, if we introduce *functions* $\lambda_j = \lambda_j(x)$ in its hypotheses, then as in 2.5

$$\tilde{F}(y) = F(y) + \sum_{j=1}^{N} \int_a^b \lambda_j(x) g_j[y(x)]\, dx,$$

and we conclude that each solution $y_0 \in \mathcal{D}$ of the differential equation for the *new* $\tilde{f}$ minimizes F on $\mathcal{D}$ [uniquely] under the pointwise constraining relations

$$g_j[y(x)] \equiv g_j[y_0(x)], \qquad j = 1, 2, \ldots, N,$$

of Lagrangian form.

Although, in general not even one such $g_j[y_0(x)]$ may be specifiable *a priori* (Why?), the vector valued version does permit minimization with given Lagrangian constraints. (See Problem 3.35 et seq.)

Corresponding applications involving inequality constraints are considered in Problem 3.31 and in §7.4.

EXAMPLE 1. To minimize

$$F(y) = \int_0^1 (y'(x))^2\, dx$$

on

$$\mathcal{D} = \{y \in C^1[0, 1]: y(0) = 0,\, y(1) = 0\},$$

when restricted to the set

$$\left\{y \in C^1[0, 1]: G(y) \stackrel{\text{def}}{=} \int_0^1 y(x)\, dx = 1\right\},$$

we observe that $f(\underline{x}, y, z) = z^2$ is strongly convex, while $g(\underline{x}, y, z) = y$ is (only) convex, on $\mathbb{R} \times \mathbb{R}^2$. Hence, we set $\tilde{f}(x, y, z) = z^2 + \lambda y$ and try to find λ for which $\lambda g(\underline{x}, y, z)$ remains convex while the differential equation

$$\frac{d}{dx} \tilde{f}_z[y(x)] = \tilde{f}_y[y(x)]$$

has a solution $y_0 \in \mathcal{D}$ for which $G(y_0) = 1$. Now since g is linear in y (and z), $\lambda g(x, y, z) = \lambda y$ is convex for each real λ. Upon substitution for $\tilde{f}$, the differential equation becomes

$$\frac{d}{dx}(2y'(x)) = \lambda \quad \text{or} \quad y''(x) = \frac{\lambda}{2},$$

which has the general solution

$$y(x) = c_1 x + c_2 + \frac{\lambda x^2}{4};$$

the boundary conditions $y(0) = 0 = c_2$ and $y(1) = 0 = c_1 + \lambda/4$ give

$$y_0(x) = \frac{-\lambda}{4} x(1 - x), \quad \text{which is in } \mathcal{D}.$$

Theorem 3.16 assures us that $y_0(x) = (-\lambda/4)x(1 - x)$ minimizes F on $\mathcal{D}$—even uniquely—under the constraint $G(y) = G(y_0)$. It remains to show that we can choose λ so that $G(y_0) = 1$ (while $\lambda g\ (\underline{x}, y, z)$ remains convex).

Thus we want

$$G(y_0) = 1 = \frac{-\lambda}{4} \int_0^1 x(1 - x)\, dx = \frac{-\lambda}{4}\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{-\lambda}{24}$$

or

$$\lambda = -24,$$

and since $-24g(\underset{\sim}{x}, y, z) = -24y$ remains convex, we have found the unique solution to our problem.

(3.18) **Remark.** In this example we can find λ to force y_0 into *any* level set of G we wish, since $\lambda g(\underset{\sim}{x}, y, z) = \lambda y$ is always convex for each value of λ. This is *not* the case in general and this approach will work only for a restricted class of level sets of G. (See Problem 3.29.)

The Hanging Cable

EXAMPLE 2 (The catenary problem). To determine the shape which a long inextensible cable (or chain) will assume under its own weight when suspended freely from its end-points at equal heights as shown in Figure 3.5, we utilize the coordinate system shown, and invoke Bernoulli's principle that the shape assumed will minimize the potential energy of the system. (See §8.3.)

We suppose the cable to be of length L and weight per unit length W, and that the supports are separated a distance $H < L$. Then *utilizing the arclength s along the cable as the independent variable*, a shape is specified by a function $y \in \mathscr{Y} = C^1[0, L]$ with $y(0) = y(L) = 0$, which has associated with it the potential energy given within an additive reference constant by

$$F(y) = W \int_0^L y(s)\, ds.$$

However, in order to span the supports, the function y must satisfy the constraining relation

$$G(y) = \int_0^L \sqrt{1 - y'(s)^2}\, ds = \int_0^L dx(s) = H,$$

where $x(s)$ denotes the horizontal displacement of the point at a distance s along the cable, since then as elementary geometry shows, $x'(s)^2 + y'(s)^2 = 1$. Clearly $|y'(s)| \le 1$ and if $|y'(s_1)| = 1$, then the cable would have a cusp at s_1.

Now $f(\underset{\sim}{s}, y, z) = Wy$ is (only) convex on $[0, L] \times \mathbb{R}^2$ while $g(\underset{\sim}{s}, y, z) = -\sqrt{1 - z^2}$ is by §3.3, Example 5, strongly convex on $[0, L] \times \mathbb{R} \times (-1, 1)$.

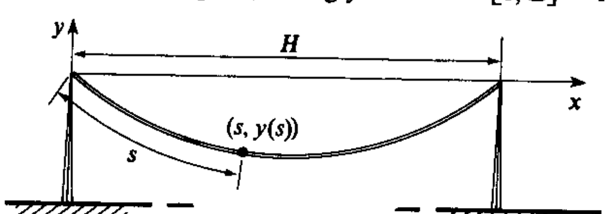

Figure 3.5

Thus by 3.11(1), the modified function $\tilde{f}(\underset{\sim}{s}, y, z) = Wy - \lambda\sqrt{1 - z^2}$ is strongly convex when $\lambda > 0$. Hence by 3.16, for $\lambda > 0$ we should seek a solution y for the differential equation

$$\frac{d}{ds} \tilde{f}_z[y(s)] = \tilde{f}_y[y(s)] \quad \text{on } (0, L)$$

which is in

$$\mathscr{D} = \{y \in C^1[0, L]: y(0) = y(L) = 0, |y'(s)| < 1, \forall\, s \in (0, L)\}.$$

Upon substitution, the differential equation becomes

$$\frac{d}{ds}\left(\frac{\lambda y'(s)}{\sqrt{1 - y'^2(s)}}\right) = W$$

or

$$\frac{\lambda y'(s)}{\sqrt{1 - y'^2(s)}} = s + c, \tag{27}$$

where we have replaced the unspecified constant λ by $W\lambda$ and introduced a new constant c.

We know that each $y \in \mathscr{D}$ which satisfies this equation for $\lambda > 0$ must be the *unique* shape sought. Hence we can make further simplifying assumptions about y if they do not preclude solution. We could, for example, suppose $y' =$ const., but it is seen that this could not solve (27). And we can suppose that y is symmetric about $L/2$, which accords with our physical intuition about the shape assumed by the cable. If we set $l = L/2$ it follows that $y'(l) = 0$, so that from (27), $c = -l$; also, we need only determine y on $[0, l]$, where we would expect that $y' \le 0$.

Thus from (27) we should have that

$$y'(s)^2 = \frac{(s - l)^2}{\lambda^2 + (s - l)^2} \quad \text{on } [0, l],$$

and so, with $y(0) = 0$, that

$$y(s) = \int_0^s \frac{(t - l)}{\sqrt{\lambda^2 + (t - l)^2}}\, dt = \sqrt{\lambda^2 + (t - l)^2}\,\Big|_0^s$$

or

$$y(s) = \sqrt{\lambda^2 + (l - s)^2} - \sqrt{\lambda^2 + l^2} \quad \text{on } [0, l]. \tag{28}$$

Now we can obviously suppose that $\lambda > 0$; however, we must satisfy the constraining relation

$$\int_0^L \sqrt{1 - y'(s)^2}\, ds = H;$$

or with our symmetry assumption, we require that

$$\int_0^l \sqrt{1 - y'(s)^2}\, ds = \frac{H}{2}.$$

Upon substitution from (28), this becomes

$$\int_0^l \sqrt{1 - \frac{(l-s)^2}{\lambda^2 + (l-s)^2}}\, ds = \int_0^l \frac{\lambda}{\sqrt{\lambda^2 + (l-s)^2}}\, ds = \frac{H}{2},$$

or with the trigonometric substitution $(l - s) = \lambda \tan \theta$, we require for $\alpha = \arctan l/\lambda$ that

$$h(\alpha) \stackrel{\text{def}}{=} \cot \alpha \int_0^\alpha \sec \theta\, d\theta = \frac{H}{2l} = \frac{H}{L}\,(<1).$$

Now, $h(\alpha) = (\log(\sec \alpha + \tan \alpha))/\tan \alpha$ is continuous and positive on $(0, \pi/2)$ and has by L'Hôpital's rule as $\alpha \searrow 0$ and $\alpha \nearrow \pi/2$, the same limits as does $\sec \alpha/\sec^2 \alpha = \cos \alpha$; viz., 1 and 0, respectively. Thus by the intermediate value theorem (§A.1), h assumes each value on $(0, 1)$ at least once on $(0, \pi/2)$. Hence $\exists\, \alpha \in (0, \pi/2)$ for which $h(\alpha) = H/L$ and for this α, $\lambda = l \cot \alpha > 0$ will provide the $y(s)$ sought.

The resulting curve is defined parametrically on $[0, l]$ by

$$y(s) = \sqrt{\lambda^2 + (l-s)^2} - \sqrt{\lambda^2 + l^2},$$
$$x(s) = \int_0^s \sqrt{1 - y'(t)^2}\, dt = \frac{H}{2} - \lambda \sinh^{-1}\left(\frac{l-s}{\lambda}\right), \tag{29}$$

which corresponds to the well-known catenary (Problem 3.30(a)).

(3.19) *Among all curves of length L joining the supports, the catenary of* (29) *will have (uniquely) the minimum potential energy and should thus represent the shape actually assumed by the cable.*

Remark. This problem is usually formulated with x as the independent variable. However, this results in an energy function which is *not* convex (Problem 3.30(b)).

Optimal Performance

EXAMPLE 3. (A simple optimal control problem). A rocket of mass m is to be accelerated vertically upward from rest at the earth's surface (assumed stationary) to a height h in time T, by the thrust (mu) of its engine. If we suppose h is so small that both m and g, the gravitational acceleration, remain constant during flight, then we wish to control the thrust to minimize the fuel consumption as measured by, say,

$$F(u) = \int_0^T u^2(t)\, dt, \tag{30}$$

for a given flight time T.

Although T will be permitted to vary later, consider first the problem in which T is fixed. We invoke Newton's second law of motion to infer that at time t, the rocket at height $y = y(t)$ should experience the net acceleration

$$\ddot{y} = u - g, \tag{31}$$

and impose the initial and terminal conditions

$$y(0) = \dot{y}(0) = 0 \quad \text{and} \quad y(T) = h.$$

Since $y(0) = 0$, then $y(T) = \int_0^T \dot{y}(t)\, dt$, so that upon subsequently integrating by parts we obtain

$$y(T) = -(T - t)\dot{y}(t)\Big|_0^T + \int_0^T (T - t)\ddot{y}(t)\, dt.$$

From (31) and the remaining boundary conditions, there follows

$$h = y(T) = \int_0^T (T - t)u(t)\, dt - \frac{gT^2}{2}.$$

Hence

$$G(u) \stackrel{\text{def}}{=} \int_0^T (T - t)u(t)\, dt = h + \frac{gT^2}{2} = k, \quad \text{say}, \tag{32}$$

and we are to minimize F on

$$\mathscr{D} = \{u \in C[0, T], u \geq 0\}$$

subject to the isoperimetric constraint (32).

According to Theorem 3.16, we introduce a constant λ and observe that the modified integrand

$$\tilde{f}(t, u, z) = u^2 + \lambda(T - t)u$$

will be strongly convex for all λ, since the second term is linear in u. Moreover, $\tilde{f}_z \equiv 0$. Thus, a $u_0 \in \mathscr{D}$ which satisfies the equation $\tilde{f}_u[u(t)] = 0 = 2u(t) + \lambda(T - t)$ and meets the constraint (32) will suffice.

on

$$\mathscr{D} = \{y \in C^1[0, \pi]: y(0) = y(\pi) = 0\},$$

when further constrained to the set where

$$\int_0^\pi y^2(x)\,dx = 1?$$

3.30. Catenary Problem. (See §3.5, Example 2.)

(a) Verify equation (29) and eliminate the parameter s to obtain the equation

$$y = \lambda \cosh\left(\frac{x-h}{\lambda}\right) - \sqrt{\lambda^2 + l^2},$$

for $0 \le x \le 2h = H$. (This is a more common representation for the catenary joining the given points.)

(b) Formulate the problem using x as the independent variable and conclude that this results in an energy function U which is not convex on $\mathscr{D} = \{y \in C^1[0, H]: y(0) = y(H) = 0\}$. (Hint: Use $v = -y$ to show that $U(y + v) - U(y)$ is not always greater than or equal to $\delta U(y; v)$ for $y, y + v \in \mathscr{D}$.)

(c)* Use the arc length s, as a parameter to reformulate the problem of finding the minimal surface of revolution (as in §1.4(a)) among all curves of *fixed length* L joining the required points. (Take $a = 0$ and $a_1 = 1 \le b_1$.)

(d) Conclude that the problem in (c) is identical to that of a hanging cable for an appropriate W, and hence there can be at most one minimizing surface. (See §3.5.)

3.31. Determine the (unique) positive function $y \in C[0, T]$ which maximizes

$$U(y) = \int_0^T e^{-\beta t} \log(1 + y(t))\,dt$$

subject to the constraint $L(y) = \int_0^T e^{-\alpha t} y(t)\,dt \le l$, where α, β, and l are positive constants. Hint: Problem 3.18, with 2.4, 3.17. (This may be given the interpretation of finding that consumption rate function y which maximizes a measure of utility (or satisfaction) U subject to a savings-investment constraint $L(y) \le l$. See [Sm], p. 80.)

3.32*. Dido's Problem.

Convexity may be used to provide partial substantiation of Dido's conjecture from Problem 1.5, in the reformulation suggested by Figure 3.8.

Verify her conjecture to the following extent:

(a) If $b > l/\pi$, prove that the function representing a circular arc (uniquely) maximizes

$$A(y) \equiv \int_{-b}^{b} y(x)\,dx$$

on

$$\mathscr{D} = \{y \in C^1[-b, b]: y(b) = y(-b) = 0\},$$

when further constrained to the l level set of $L(y) \equiv \int_{-b}^{b} \sqrt{1 + y'(x)^2}\,dx$.

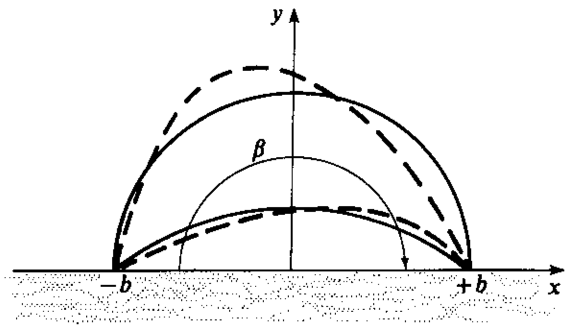

Figure 3.8

(b) If $b = l/\pi$, show that the function representing the semicircle accomplishes the same purpose for a suitably chosen $\mathscr{D}^*$ (see Problem 3.21).

(c)* In parts (a) and (b), compute the maximal area as a function of β, the angle subtended by the arc; show that this function increases with β on $(0, \pi]$.

(d) Why does this not answer the problem completely? Can you extend the analysis to do so?

3.33. Let I be an interval in $\mathbb{R}$ and D be a domain in $\mathbb{R}^{2d}$. For $x \in \mathbb{R}$, and $Y = (y_1, \ldots, y_d)$, $Z = (z_1, \ldots, z_d) \in \mathbb{R}^d$ a function $f(\underset{\sim}{x}, Y, Z)$ is said to be [strongly] convex on $I \times D$ if $f, f_Y = (f_{y_1}, \ldots, f_{y_d})$, and $f_Z = (f_{z_1}, \ldots, f_{z_d})$ are defined and continuous on this set and satisfy the inequality

$$f(x, Y + V, Z + W) - f(x, Y, Z) \ge f_Y(x, Y, Z) \cdot V + f_Z(x, Y, Z) \cdot W,$$
$$\forall\,(x, Y, Z), (x, Y + V, Z + W) \in I \times D$$

[with equality at (x, Y, Z) only if $V = \mathcal{O}$ or $W = \mathcal{O}$].

(a) Show that if $f(\underset{\sim}{x}, Y, Z)$ is [strongly] convex on $[a, b] \times \mathbb{R}^{2d}$, then

$$F(Y) = \int_a^b f[Y(x)]\,dx = \int_a^b f(x, Y(x), Y'(x))\,dx$$

is [strictly] convex on

$$\mathscr{D} = \{Y \in (C^1[a, b])^{2d}: Y(a) = A,\ Y(b) = B\},$$

where $A, B \in \mathbb{R}^d$ are given.

(b) If $f(\underset{\sim}{x}, Y, Z)$ is strongly convex on $[a, b] \times \mathbb{R}^{2d}$, then prove that each $Y \in \mathscr{D}$ which satisfies the vector differential equation $(d/dx) f_Z[Y(x)] = f_Y[Y(x)]$ (i.e.; $(d/dx) f_{z_j}[Y(x)] = f_{y_j}[Y(x)]$, $j = 1, 2, \ldots, d$) on (a, b) is the unique minimizing function for F on $\mathscr{D}$.

3.34. Use the results in Problem 3.33 to formulate and prove analogous vector valued versions of: (a) Theorem 3.7, (b) Corollary 3.8, and (c) Proposition 3.9.

3.35. (a) Formulate and prove a vector valued version of Theorem 3.16.

(b) Modify Theorem 3.16 to cover the case of a single Lagrangian constraint, and prove your version. Hint: Proposition 2.5 with 3.17.

(c) Formulate a vector valued version of the modified problem in (b) with both isoperimetric and Lagrangian constraints.

(f) When $\rho_t = \mu_t = 0$, use the total energy function $E(t) = T + U$ to give a uniqueness result modelled after Proposition 8.4 as extended in Problem 8.17.

(g) Study the expression in (d) to obtain alternate sets of boundary conditions which would lead to the same differential equation as in (e).

(h) When $\rho = \mu =$ const. and $p = 0$, determine an ordinary differential equation for $u_0 = u_0(x)$, if $u(t, x) \equiv u_0(x) \cos \omega t$ is to be a solution of (75).

(i) Can you find or guess a nontrivial solution u_0 of the equation in (h) which will permit u to meet the cantilever conditions in (e) at least for certain values of ω? Hint: See §6.6.

8.21*. Transverse Motion of a Uniform Plate. If the membrane discussed in §8.9(b) is replaced by a plate of uniform thickness and material, we may neglect the energy of stretching in comparison with that of bending which is now given (approximately) by

$$U = \frac{\mu}{2} \int_D [(u_{xx}^2 + u_{yy}^2) - 2(1 - \tau)(u_{xx}u_{yy} - u_{xy})^2] \, dX, \tag{76}$$

where, of course, $u(t, X)$ denotes the vertical position of a center section at time t, and μ and $\tau < 1$ are positive material *constants.*

(a) Argue that for an appropriate constant density ρ, the kinetic energy of motion at time t should be approximately $T = \frac{1}{2} \int_D \rho u_t^2 \, dX$.

(b) Set $A(u) = \int_0^b (T - U) \, dt$, and, neglecting external loading, reason that for *some* boundary conditions, stationarity of A at u in C^4 requires that u should satisfy the equation

$$\rho u_{tt} + \mu \Delta^2 u = 0, \tag{77}$$

where

$$\Delta^2 u = \Delta(\Delta u) = \Delta(u_{xx} + u_{yy}).$$

(c) Which equation is $u_0 = u_0(X)$ required to satisfy in order that $u(t, X) = u_0(X) \cos \omega t$ be a solution of (77)?

(d) For static equilibrium of the loaded plate with pressure $p = p(X)$, when all functions are time independent, use convexity of the integrand of

$$\tilde{U} \stackrel{\text{def}}{=} U - \int_D pu \, dX$$

to conclude that even for a nonplanar plate, only *stable* equilibrium is possible, and it is uniquely characterized by a u_0 which satisfies the equation:

$$\mu \Delta^4 u = p.$$

Hint: The term $u_{xx}u_{yy} - u_{xy}^2 = \operatorname{div}(u_x u_{yy}, -u_x u_{xy})$.

CHAPTER 9*

Sufficient Conditions for a Minimum

As we have noted repeatedly, the equations of Euler–Lagrange are necessary but not sufficient to characterize a minimum value for the integral function

$$F(Y) = \int_a^b f(x, Y(x), Y'(x)) \, dx = \int_a^b f[Y(x)] \, dx$$

on a set such as

$$\mathscr{D} = \{Y \in C^1([a, b])^d : Y(a) = A, \, Y(b) = B\},$$

since they are only conditions for the stationarity of F. However, in the presence of [strong] convexity of $f(\underline{x}, Y, Z)$ these conditions do characterize [unique] minimization. [Cf. §3.2, Problem 3.33 et seq.] Not all such functions are convex, but we have also seen in §7.6 that a minimizing function Y_0 must necessarily satisfy the Weierstrass condition $\mathscr{E}(x, Y_0(x), Y_0'(x), W) \geq 0$, $\forall \, W \in \mathbb{R}^d$, $x \in [a, b]$, where

$$\mathscr{E}(x, Y, Z, W) \stackrel{\text{def}}{=} f(x, Y, W) - f(x, Y, Z) - f_Z(x, Y, Z) \cdot (W - Z), \tag{1}$$

and this is recognized as a convexity statement for $f(x, Y, Z)$ along a trajectory in $\mathbb{R}^{2d+1}$ defined by Y_0.

This chapter is devoted to showing that conversely, when $f(\underline{x}, \underline{Y}, Z)$ is [strictly] convex (§9.2) in the presence of an appropriate *field*, then each stationary Y_0 in $\mathscr{D}$ does minimize F on $\mathscr{D}$ [uniquely] (§9.3, §9.4) and this will afford a solution for the brachistochrone problem. The method extends in principle to problems with variable end point conditions (§9.4) and to those such as the classical isoperimetric problem on which constraints are imposed (§9.5).

However, the field in question requires an entire family of stationary functions with special properties which may, or may not, exist. A central field will suffice (§9.6) and this provides further insight into the problem of finding the minimal surface of revolution. In §9.7 we provide conditions which assure that a given stationary trajectory Γ_0 may be considered as that of a central field. We encounter a new criterion, that of Jacobi, but in §9.9 we demonstrate that it, too, is almost essential for even local minimization. This embedding of Γ_0 together with the appropriate convexity of f supplies sufficient conditions for the local minimization of integral functions F, both in the weak and in the strong senses (§9.8).

To motivate the inquiry, we examine first in §9.1 the original method used to attack such problems.

§9.1. The Weierstrass Method

In his lectures of 1879, Weierstrass presented the following approach to prove that a given stationary function $Y_0 \in (C^1[a, b])^d$ minimizes

$$F(Y) = \int_a^b f[Y(t)]\,dt = \int_a^b f(t, Y(t), Y'(t))\,dt$$

on

$$\mathcal{D} = \{Y \in (C^1[a, b])^d : Y(a) = Y_0(a);\ Y(b) = Y_0(b)\},$$

(where for simplicity we suppose that $f \in C^1([a, b] \times \mathbb{R}^{2d})$.

Let Y in $\mathcal{D}$ be a competing function and assume that each $x \in (a, b]$ determines a *unique* function $\Psi(\cdot; x)$, stationary for f on (a, x) as in §6.7, whose graph joins $(a, Y_0(a))$ to $(x, Y(x))$ as shown in Figure 9.1.

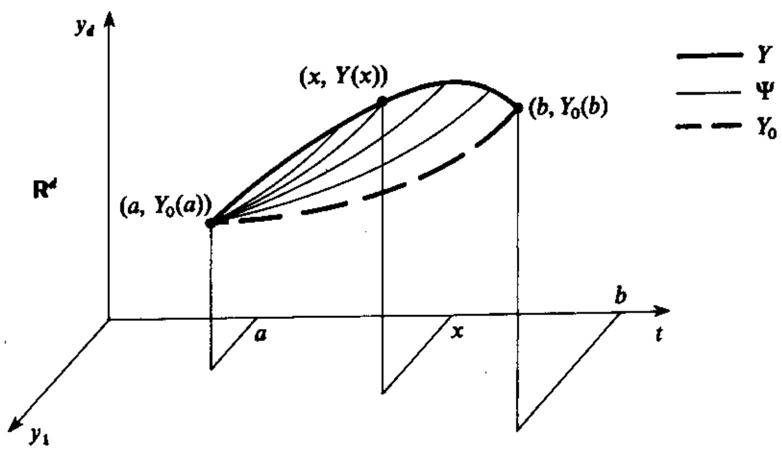

Figure 9.1

Then, in particular, $\Psi(t; b) = Y_0(t)$, (Why?), and we consider the integral function

$$\sigma(x) \stackrel{\text{def}}{=} -\int_a^x f[\Psi(t; x)]\,dt - \int_x^b f[Y(t)]\,dt, \tag{2}$$

which interpolates between

$$\sigma(a) = -\int_a^b f[Y(t)]\,dt = -F(Y)$$

and

$$\sigma(b) = -\int_a^b f[\Psi(t; b)\,dt = -\int_a^b f[Y_0(t)]\,dt = -F(Y_0),$$

so that $F(Y) - F(Y_0) = \sigma(b) - \sigma(a)$. Were $\sigma'(x) \geq 0$, it would follow by the mean value theorem that $F(Y) - F(Y_0) \geq 0$. Moreover, if also σ' is continuous on $[a, b]$, then equality holds *iff* $\sigma' \equiv 0$. (§A.1, §A.2.)

For example, when $d = 1$, consider the problem of minimizing the (nonconvex) function

$$F(y) = \int_0^b [y'(t)^2 - y(t)^2]\,dt$$

on

$$\mathcal{D} = \{y \in C^1[0, b] : y(0) = y(b) = 0\} \quad \text{for } b < \pi.$$

Here, the stationary functions ψ satisfy the Euler–Lagrange equation

$$\frac{d}{dt} 2\psi'(t) = -2\psi(t) \quad \text{or} \quad \psi''(t) + \psi(t) = 0,$$

with the well-known general solution

$$\psi(t) = c_0 \cos t + c_1 \sin t.$$

Since $b < \pi$, the only solution in $\mathcal{D}$ is $y_0 = \mathcal{O}$, and for a given y in $\mathcal{D}$, it is seen by inspection that for each $x \in (0, b)$:

$$\psi(t; x) \stackrel{\text{def}}{=} y(x)\frac{\sin t}{\sin x}$$

is the unique function which is stationary for f and satisfies $\psi(0; x) = 0$ with $\psi(x; x) = y(x)$. (See Figure 9.2.)

Thus, for this example, equation (2) becomes

$$\sigma(x) = -\int_0^x [\psi'(t; x)^2 - \psi(t; x)^2]\,dt - \int_x^b [y'(t)^2 - y(t)^2]\,dt$$

$$= -\frac{y^2(x)}{\sin^2 x}\int_0^x (\cos^2 t - \sin^2 t)\,dt - \int_x^b [y'(t)^2 - y(t)^2]\,dt,$$

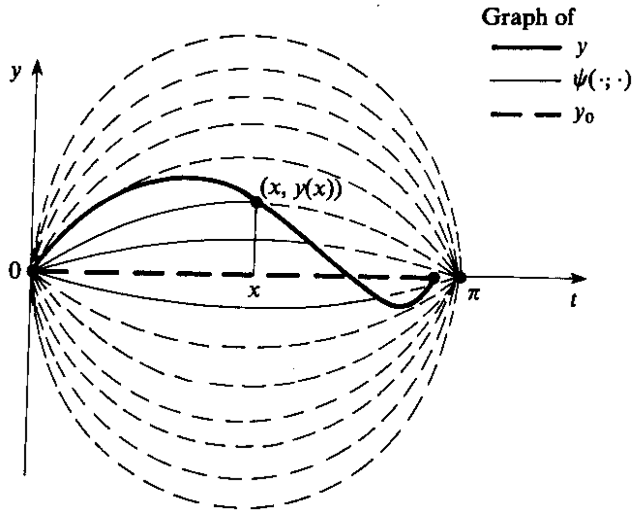

Figure 9.2

so that after integration,

$$\sigma(x) = -y^2(x)\cot x - \int_x^b [y'(t)^2 - y(t)^2]\,dt. \qquad (3)$$

Then by the fundamental theorem of calculus, for $x \in (0, b)$:

$$\begin{aligned}\sigma'(x) &= -2y(x)y'(x)\cot x + y^2(x)\operatorname{cosec}^2 x - y^2(x) + y'(x)^2\\ &= (y(x)\cot x - y'(x))^2 \geq 0,\end{aligned}$$

with equality *iff* $y'(x) - y(x)\cot x = 0$.

Now, an apparent difficulty occurs when $x = 0$ as a result of the multiplicity of stationary functions ψ which pass through (0, 0); namely, that $\sigma(0)$ is *not* defined by (3). However, by L'Hôpital's rule,

$$\lim_{x \searrow 0} \frac{y^2(x)}{\sin x} = \lim_{x \searrow 0} \frac{2y(x)y'(x)}{\cos x} = 0,$$

for y in $\mathscr{D}$, so that $\sigma(0+) = -F(y)$. Thus $F(y) - F(y_0) = \sigma(b) - \sigma(0+) \geq 0$; or $F(y) \geq F(y_0) = 0$, with equality *iff* $y'(x) = y(x)\cot x$, $\forall\, x \in (0, b)$. But for y in $\mathscr{D}$, this implies that $y'(b) = y(b) = 0$ (since $b < \pi$), and hence that $y(x) \equiv 0 = y_0(x)$.

Thus we have proven that for $b < \pi$, $y_0 = \mathcal{O}$ is the unique minimizing function for F on $\mathscr{D}$. When $b = \pi$, the method is still applicable, (and provides a proof for Dido's conjecture of Problem 1.5), but the minimizing function y_0 is no longer unique. However, when $b > \pi$, y_0 fails to minimize. (See Problem 9.1.)

Having illustrated the effectiveness of the Weierstrass approach in a simple case, we return to the general problem and equation (2). As defined, $\Psi(t; x)$ depends on two variables, and as above, we use the prime to denote t differentiation. Supposing that Ψ is C^2, it follows that

$$(\Psi')_x = \Psi_{tx} = \Psi_{xt} = (\Psi_x)'. \qquad (4)$$

Then, from Leibniz' rule (A.14) applied to (2), we obtain

$$\sigma'(x) = f[Y(x)] - f[\Psi(x; x)] - \int_a^x \frac{\partial}{\partial x} f[\Psi(t; x)]\,dt, \qquad (5)$$

and the integrand is from the chain rule, (4), and stationarity, given by

$$\begin{aligned}\frac{\partial}{\partial x} f(t, \Psi(t; x), \Psi'(t; x)) &= f_Y[\Psi(t; x)]\Psi_x(t; x) + f_Z[\Psi(t; x)](\Psi')_x(t; x)\\ &= \frac{\partial}{\partial t}\{f_Z[\Psi(t; x)] \cdot \Psi_x(t; x)\}.\end{aligned}$$

But $\Psi(x; x) \equiv Y(x)$ by construction, so that $\Psi_x(x; x) = Y'(x) - \Psi'(x; x)$ (Why?); while $\Psi_x(a; x) = 0$, since $\Psi(a; x) = Y_0(a)$, is constant. Hence, after integration and substitution, (5) becomes

$$\begin{aligned}\sigma'(x) = {} & f(x, Y(x), Y'(x)) - f(x, Y(x), \Psi'(x; x))\\ & - f_Z(x, Y(x), \Psi'(x; x)) \cdot (Y'(x) - \Psi'(x; x)),\end{aligned}$$

or upon utilizing the definition (1),

$$\sigma'(x) = \mathscr{E}(x, Y(x), \Psi'(x; x), Y'(x)). \qquad (6)$$

(For $d = 1$, the reader should verify each step of this derivation by purely formal calculations.) Thus finally, we obtain *Weierstrass' formula*:

$$F(Y) - F(Y_0) = \int_a^b \sigma'(x)\,dx = \int_a^b \mathscr{E}(x, Y(x), \Psi'(x; x), Y'(x))\,dx, \qquad (7)$$

which proves that $\mathscr{E} \geq 0$ will imply that $F(Y) \geq F(Y_0)$, provided that an appropriate family of stationary functions $\Psi(\cdot; \cdot)$ having *all* of the assumed properties is available. Unfortunately, it is quite difficult to prescribe conditions which ensure the existence of such families (one for each competing $Y \in \mathscr{D}$), and instead in §9.3 et seq. we shall concentrate on a less direct approach of Hilbert, which yields Weierstrass' result even for piecewise C^1 functions $\hat{Y}$.

(Problems 9.1–9.2, 9.5–9.6)

§9.2. [Strict] Convexity of $f(\underline{x}, \underline{Y}, Z)$

The definition (1) of the Weierstrass excess function for a given $f = f(x, Y, Z)$; viz.,

$$\mathscr{E}(x, Y, Z, W) = f(x, Y, W) - f(x, Y, Z) - f_Z(x, Y, Z) \cdot (W - Z),$$

and our wish to consider $\mathscr{E} \geq 0$, suggests in comparison with 3.4, the following:

(9.1) Definition. $f(\underline{x}, \underline{Y}, Z)$ is said to be [strictly] *convex* in a set $S \subseteq \mathbb{R}^{2d+1}$, when f and f_Z are defined and continuous in S and satisfy the inequality $\mathscr{E}(x, Y, Z, W) \geq 0$, or equivalently,

$$f(x, Y, W) - f(x, Y, Z) \geq f_Z(x, Y, Z) \cdot (W - Z), \tag{8}$$

when $(x, Y, Z) \in S$ and $(x, Y, W) \in S$, [with equality at (x, Y) *iff* $W = Z$].

Usually, the set S will be of the form $S = D \times \mathbb{R}^d$ for a *domain* $D \subseteq \mathbb{R}^{d+1}$.

Since the [strict] convexity of $f(\underline{x}, \underline{Y}, Z)$ is identical with the [strong] convexity of $f(\underline{x}, \underline{Y}, Z)$ as defined in Chapter 3 (Problem 3.33), the usual linear combinations of such functions remain [strictly] convex. See Proposition 3.2 and Facts 3.11.

EXAMPLE 1. $f(\underline{x}, Y, Z) = -|Y|^2 + |Z|^2$ is not convex, but $f(\underline{x}, \underline{Y}, Z)$ is strictly convex in $\mathbb{R}^{2d+1}$ since $f_Z(x, Y, Z) = 2Z$ so that

$$\begin{aligned} f(x, Y, W) - f(x, Y, Z) &= |W|^2 - |Z|^2 = (W + Z) \cdot (W - Z) \\ &= |W - Z|^2 + 2Z \cdot (W - Z) \\ &\geq f_Z(x, Y, Z) \cdot (W - Z), \end{aligned}$$

with equality at (x, Y) *iff* $|W - Z|^2 = 0$ or $W = Z$.

EXAMPLE 2. For $d = 1$, the brachistochrone function

$$f(\underline{x}, y, z) = \sqrt{\frac{1 + z^2}{y}} \quad \text{(of §1.2(a))}$$

is not convex, but $f(\underline{x}, \underline{y}, z)$ is strictly convex for the half space

$$\{(x, y, z) \in \mathbb{R}^3 : y > 0\}$$

since $f_{zz}(x, y, z) > 0$. (See Proposition 3.10 and the next example.)

EXAMPLE 3. For $d = 2$, when $Y = (x, y)$, the function $f(\underline{t}, \underline{Y}, Z) = \sqrt{y}|Z|^2$ is [strictly] convex on the half-space

$$\{(t, x, y, Z) \in \mathbb{R}^5 : y \geq 0\}, [\{(t, x, y, Z) \in \mathbb{R}^5 : y > 0\}].$$

For, by the computation of Example 1,

$$\begin{aligned} f(t, Y, W) - f(t, Y, Z) &= \sqrt{y}(|W - Z|^2 + 2Z \cdot (W - Z)) \\ &\geq f_Z(t, Y, Z) \cdot (W - Z), \end{aligned}$$

[with equality for $y > 0$ *iff* $W = Z$].

EXAMPLE 4. For $d = 1$, the function $f(\underline{x}, \underline{y}, z) = -\sqrt{(1 - z^2)}/\underline{y}$ is strictly convex on the set $S = \{(x, y, z) \in \mathbb{R}^3 : y > 0, |z| < 1\}$. (See Example 5 of §3.3, and Example 2 above.)

From §0.13, there is the following generalization of Proposition 3.10:

(9.2) Proposition. *If $f = f(x, Y, Z)$ together with its partials f_{z_i} and $f_{z_i z_j}$, $i, j = 1, 2, \ldots, d$ is continuous in a Z-convex set $S \subseteq \mathbb{R}^{2d+1}$ (one which contains the segment joining each pair of its points (x, Y, Z_0), (x, Y, Z_1)) and the matrix f_{ZZ} is positive semidefinite* [positive definite] *in S, then $f(\underline{x}, \underline{Y}, Z)$ is* [strictly] *convex in S.*

PROOF. For (x, Y, Z_0) and (x, Y, Z_1) in S and $t \in [0, 1]$, the point

$$Z_t \stackrel{\text{def}}{=} (1 - t)Z_0 + tZ_1$$

lies on a segment contained in S by hypothesis. Integrating by parts, we get

$$\begin{aligned} f(x, Y, Z_1) - f(x, Y, Z_0) &= \int_0^1 \frac{d}{dt} f(x, Y, Z_t)\, dt \\ &= (Z_1 - Z_0) \cdot \int_0^1 f_Z(x, Y, Z_t)\, d(t - 1) \\ &= (Z_1 - Z_0) \cdot (t - 1) f_Z(x, Y, Z_t) \Big|_{t=0}^{t=1} \\ &\quad + \int_0^1 (1 - t)\left(\sum_{i,j=1}^{d} f_{z_i z_j}(x, Y, Z_t) v_i v_j\right) dt, \end{aligned}$$

where $V = Z_1 - Z_0$. The last term is nonnegative when $Z_1 \neq Z_0$ as a consequence of the assumed semidefiniteness of f_{ZZ} [and with positive definiteness of f_{ZZ}, it vanishes *iff* $V = \mathcal{O}$]. Hence

$$f(x, Y, Z_1) - f(x, Y, Z_0) \geq f_Z(x, Y, Z_0) \cdot (Z_1 - Z_0)$$

[with equality *iff* $Z_1 = Z_0$]. □

A (spherical) neighborhood of a point (x_0, Y_0, Z_0) is Z-convex. If D is a domain in $\mathbb{R}^{d+1}$, then $S = D \times \mathbb{R}^d$ is Z-convex.

§9.3. Fields

The Weierstrass construction in §9.1, when possible, results in a *family* of *stationary trajectories* (the graphs of the functions $\Psi(\cdot\,; x)$) which is consistent in that one and only one member of the family passes through a given point $(x, Y(x))$. Suppose more generally, that for a given f we have a single family of stationary functions whose trajectories cover a domain $D \subseteq \mathbb{R}^{d+1}$ *consistently* in that through each point $(x, Y) \in D$ passes one and only one trajectory of the family, say that represented by $\Psi(\cdot\,; (x, Y)) \in (C^1[a, b])^d$. Then the *direction* of the *tangent line* to the *trajectory at* (x, Y) given by

$$\Phi(x, Y) \stackrel{\text{def}}{=} \Psi'(x; (x, Y))$$

determines a vector valued function in D (and one, moreover, whose values are required for Weierstrass' formula (7) along each competing trajectory).