EXPOSITIONES
MATHEMATICAE

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Catenaria Vera - The True Catenary

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Invenire, quam curvam referat funis laxus & inter duo puncta fixa libere suspensus. James Bernoulli (Acta Eruditorum, 1690)

Abstract

We give an explicit solution of the catenary problem in a $-1/r$ potential, including full discussion of the boundary value problem arising and of the existence question (which is more delicate than in other classical variational problems and is therefore somewhat neglected in the textbooks even for the classical catenary). Numerical examples illustrate the results. While reviewing some history, we also attempt a style that makes the discussion feasible for teaching purposes, in particular by bridging the gap between classical textbooks and the more modern tools. The appearance of the logarithmic spiral as a solution is a nice surprise in addition. Our existence proof by direct methods, enhanced with a priori estimates, generalizes to other central potentials.

0 The problem and its history

When James Bernoulli put forward the challenge of finding the curve describing the shape of a(n idealized) chain consisting of equal (infinitesimal) links, fixed at both ends and hanging under the sole influence of gravitation, most scientists believed in the parabola as the appropriate model. They could rely on the authority of Galilei, who in *Dialogo Secondo* of his *Discorsi* ... (1638) [6, p. 146] held the position that *questa catenella si piega in figura Parabolica.*

Let us, for the moment, assume the chain to be fixed in a cartesian system with horizontal x-axis and vertical y-axis in two points of equal *height* $h > 0$ above the x-axis at a distance $2d$ ($d > 0$ is called the *dimension* of the chain) from each other, i.e. in the points $(-d, h)$ and (d, h), say. The *length* $2l$ of the chain necessarily fulfils $l \geq d$ (cf. Figure 1).

For the *Galileian parabola* we then have with a parameter $g \geq 0$:

$$\forall x \in [-d, d] : y(x) = h - g(d^2 - x^2),$$

whence with $\gamma := 2dg$:

$$\forall x \in [-d, d] : \frac{y(x)}{d} = \frac{h}{d} - \frac{\gamma}{2}\left(1 - (\frac{x}{d})^2\right), \tag{1}$$

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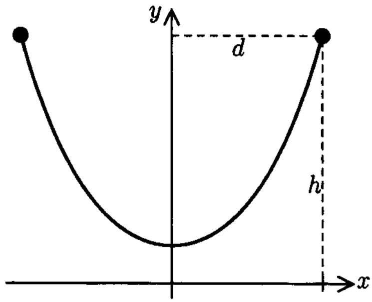


Figure 1: Chain in cartesian coordinates

and $\gamma \geq 0$ is uniquely determined by l through

$$\frac{2l}{d} = \frac{2}{d}\int_0^d \sqrt{1+y'(\xi)^2}\,d\xi = \frac{2}{\gamma}\int_0^\gamma \sqrt{1+\tau^2}\,d\tau = \sqrt{1+\gamma^2} + \frac{1}{\gamma}\mathrm{Arsh}(\gamma).$$

(By looking at the last but one expression we see that the r.h.s. is strictly monotone increasing and goes to 2 as $\gamma \to 0$, so that all admissible lengths are covered.)

In a passage of *Dialogo Qvarto** of the *Discorsi* [6, p. 284], Galilei considers the parabola as an approximation to the shape of a hanging chain (*... le quali assai si auuicinano alle paraboliche ...*), which is better (*... la catenella camina quasi* ad unguem *sopra la parabola.*) if the chain is flat, i.e. the tension is high. In fact, a few years later, in 1646, in a letter to the French mathematician Marin Mersenne, the 17-years-old Christiaan Huygens comes to the conclusion [7, p. 37]

Nulla ergo catena pendet secundum lineam parabolicam.

(Therefore, no chain hangs according to the parabolic line.) However, he is not able to determine the true shape of the curve. But once infinitesimal methods in analysis were available, James Bernoulli's question was immediately taken up in the following issue of *Acta Eruditorum* (1691), by his brother John Bernoulli, Leibniz, and, now aged 62, Huygens. Leibniz, for instance, observes (Figure 2, cf. [10, Tab. VII ad p. 278]) that $\overline{BC} = \mathrm{Arsh}(\widehat{AC})$, leading to $y(x) = \cosh(x)$ under the further assumption $\overline{OA} = 1$.

More generally we have in our notation

$$\forall x \in [-d,d] : \frac{y(x)}{d} = \frac{h}{d} - \frac{1}{\beta}\left(\cosh(\beta) - \cosh(\beta\frac{x}{d})\right), \tag{2}$$

and the parameter $\beta \geq 0$ is uniquely determined by

$$\frac{l}{d} = \mathrm{sinch}(\beta),$$

*We adhere to the orthography of the source, throughout.

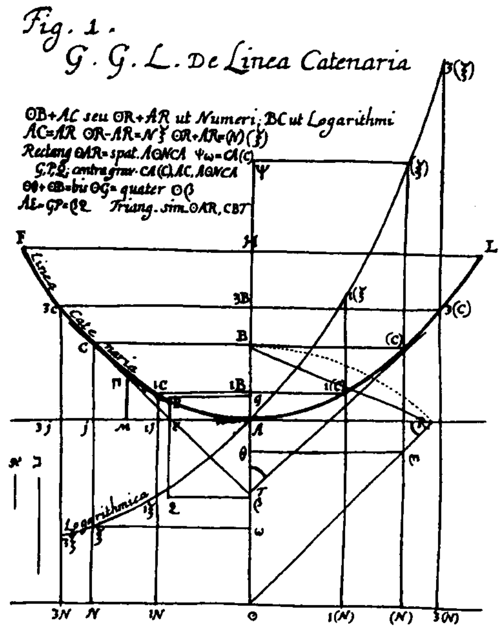


Figure 2: *Linea catenaria* of Leibniz (Photo: Deutsches Museum, München)

where the function sinch (for *sinus cardinalis hyperbolicus*, a designation adopted in analogy to the sinc-function; cf. [16, p. 41]) is defined by

$$\forall \beta \in \mathbb{R} : \operatorname{sinch}(\beta) = \sum_{n=0}^{\infty} \frac{\beta^{2n}}{(2n+1)!} \left(= \frac{\sinh(\beta)}{\beta} \right) . \tag{3}$$

(Note that this function is 1 for $\beta = 0$ and tends to ∞ as $\beta \to \infty$; its derivative is 0 if and only if $\beta = 0$.) Leibniz therefore calls the hyperbolic cosine *Linea catenaria*, the *catenary.*

It is obvious that the shape of the curve as given by β is determined already by the quotient of the geometric quantities l and d, the height h just leading to a shift in direction of the y-axis. In particular, no physical quantities like the linear mass density of the chain or the value of gravitational force enter into the formula. This force was assumed constant and in the direction of the negative y-axis. If, however, we consider a variable gravitational field centered in 0, the chain cannot cross the line between a suspension point and the origin, a fact which becomes obvious if we rotate this line to an upright position. (Cf. the problem of astronauts who wanted to erect the U.S. flag on the moon; see below for a more serious argument.) Since (from (2))

$$y'(x) = \sinh(\beta \frac{x}{d}) ,$$

$y(0) = h - \frac{d}{\beta}(\cosh(\beta) - 1)$ is a minimum and $y'(d) = \frac{\beta l}{d}$, whence $y(0) \leq 0$ or $y'(d) > \frac{h}{d}$

for h sufficiently small, so that the curve would leave the admissible region. We therefore may state:

Nulla ergo catena pendet secundum lineam catenariam.

But what is the *Catenaria vera*, the true catenary? John Bernoulli dealt with this problem in his *Solutio problematis catenarii generaliter concepti* (Opus CLXXIII), but he left a crucial parameter undetermined (see [2, p. 234-241]; cf. also [8, p. 155-162]). We will give a complete answer to the question in the following, together with a comprehensive study of the corresponding upright arches, in the course of which we will not only show that the statement of Robert Hooke (1675) (cf. [15, p. xxi]),

Ut pendet continuum flexile, sic stabit contiguum rigidum inuersum.

(Just like the flexible line is hanging, so stands inverted the rigid arch.) is not quite correct, but also that arches exhibit some surprising special cases.

1 The model

The mathematical model of an ideal homogeneous chain fixed at its ends in two suspension points P_1 and P_2 and subject only to a gravitational force centered at the origin 0 is a rectifiable curve of prescribed length $2l$ from P_1 to P_2 minimizing potential energy (or in other words possessing the lowest center of gravity, as James Bernoulli pointed out; this is known as *Bernoulli's principle*). This is because otherwise the exceeding potential energy could be converted into kinetic energy, leading to a deformation of the chain. First of all we have to convince ourselves that such a curve $t \mapsto \gamma(t)$ automatically lies in the plane spanned by P_1, P_2 and 0, and that the distance $r = |\gamma(t)|$ of its points from the origin is a function of the angle φ at 0. This will allow us to write the potential energy $-\int_{t_1}^{t_2} \frac{1}{|\gamma(t)|}|\dot{\gamma}(t)|\,dt$ as the functional $r \mapsto -\int_{\varphi_1}^{\varphi_2} \frac{1}{r(\varphi)}\sqrt{r(\varphi)^2 + \dot{r}(\varphi)^2}\,d\varphi$. (Derivatives w.r.t. angles are denoted by dots.) We give a sketch of the argument.

For a minimizing curve $\gamma : t \mapsto \gamma(t) \in \mathbb{R}^3$, the function $t \mapsto |\gamma(t)|$ is either monotonic or else has a single minimum. For otherwise, there exist $t_1 < t_2$ such that $|\gamma(t_1)| = |\gamma(t_2)| \leq |\gamma(t)|$ for all $t \in [t_1, t_2]$. Let Π be the plane through $\gamma(t_1)$ and $\gamma(t_2)$ whose normal lies in the span of $\gamma(t_1)$ and $\gamma(t_2)$. Reflecting $\gamma|_{[t_1,t_2]}$ with respect to Π yields a curve of the same length, but of lower potential energy.

Given any rectifiable curve in $\mathbb{R}^3$ between P_1 and P_2 and satisfying the monotonicity property just established, we construct a curve $\tilde{\gamma}$ between P_1 and P_2 with the same length, but lying in the plane spanned by P_1, P_2 and 0, and with strictly smaller potential energy than γ (except if γ lay already in that plane, in which case $\tilde{\gamma} = \gamma$). Indeed, let γ' be the projection of γ into that plane, and attach a vertical piece of the appropriate length (and to be traversed back and forth) to the lowest point of γ'. This results in the curve $\tilde{\gamma}$ with the promised properties (first picture in Figure 3). This shows that we may restrict ourselves to planar curves from the beginning.

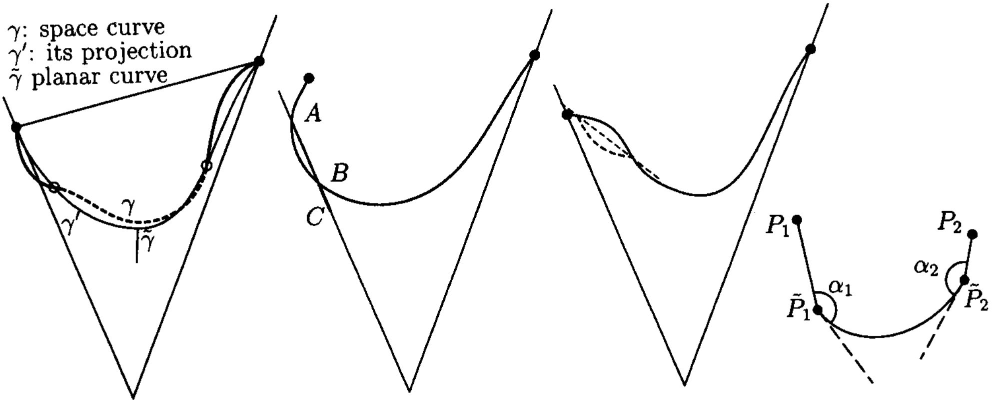


Figure 3: Reducing the potential energy of a chain

We next show that minimizing planar curves $t \mapsto \gamma(t) = (r(t), \varphi(t))$ have to be graphs of a function $\varphi \mapsto r(\varphi)$. Firstly, $t \mapsto \varphi(t)$ has to be monotonic, because otherwise a similar modification from γ to $\tilde{\gamma}$ would again lower the potential energy. (In the second picture of Figure 3, replace the arc $\widehat{AB}$ by the segments $\overline{AC}$ and $\overline{CB}$.) Secondly, a minimizing curve γ must be convex, because otherwise a reflection would lower the energy (third picture). These two properties already make γ a graph of a function $\varphi \mapsto r(\varphi)$ except that vertical pieces may be attached to the ends, as in the fourth picture.

If such pieces actually occur, the angles $\alpha_{1/2}$ are defined because the curve is convex. If, say $\alpha_1 < \pi$, we show that the curve can be changed in an ε–neighbourhood of $\tilde{P}_1$ such as to decrease the potential energy. The linear approximation within this neighbourhood is sufficient, and it consists of two straight segments in a force field which can be considered to be constant. Letting the corner point joining these segments vary on the appropriate ellipse, it is a simple calculus exercise to see that the derivative of the potential energy in the direction of constant length does not vanish. This shows that no corners can occur at $\tilde{P}_{1/2}$, or elsewhere, for that matter.

Now if the curve γ minimizes potential energy from P_1 to P_2, then so does its segment between $\tilde{P}_1$ and $\tilde{P}_2$ within its class of competitors. We will show below that minimizing curves of the form $(\varphi, r(\varphi))$ do not have vertical tangents. Therefore no vertical pieces could have been attached in a minimizing curve, because this would result in corners.

So we may consider polar coordinates $(r, \varphi) \in]0, \infty[\times [\varphi_1, \varphi_2]$ given by $x = r\sin(\varphi)$ and $y = r\cos(\varphi)$. (The slight deviation from convention is motivated by our experience that gravitational attraction comes from below.) Then $P_1 = (r_1, \varphi_1)$, $P_2 = (r_2, \varphi_2)$, and the length $2l$ are given (see Figure 4), and we have the geometric restriction

$$0 < \Phi := \frac{\varphi_2 - \varphi_1}{2} < \frac{\pi}{2} \tag{4}$$

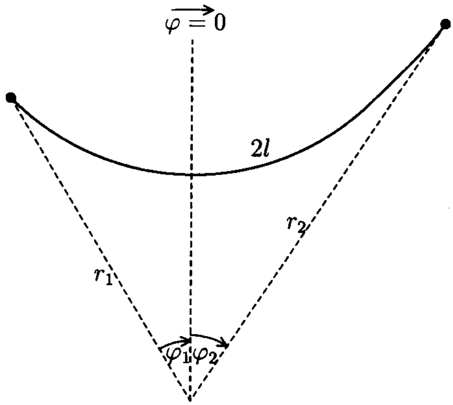


Figure 4: Chain in polar coordinates

as well as the physical restriction

$$\left(\frac{r_2 - r_1}{2}\right)^2 + r_1 r_2 \sin(\Phi)^2 < l^2 < \left(\frac{r_2 + r_1}{2}\right)^2 . \tag{5}$$

(Equality on the l.h.s. corresponds to the taut chain by virtue of the cosine rule; the right inequality keeps the chain off the gravitational center.) The original quantities d and h of Section 0 are therefore to be replaced by $\Phi = \arctan(\frac{d}{h})$ and $r_1 = r_2 = r_\Phi := r(\Phi) = \sqrt{d^2 + h^2}$.

To represent the potential energy of the hanging chain, we sum over arbitrarily small links between ψ_n and ψ_{n+1}, $n \in \{0, \ldots, N-1\}$ with $\psi_0 = \varphi_1, \psi_N = \varphi_2, N \in \mathbb{N}$, such that with $p(\psi) := (r(\psi)\sin(\psi), r(\psi)\cos(\psi))$ we get for the inverse square gravitational law of Hooke and Newton (G is the universal gravitational constant, M the mass of the central body, μ the linear mass density of the chain):

$$\begin{aligned}
&\lim_{N\to\infty}\left(-\sum_{n=0}^{N-1} GM\mu \frac{|p(\psi_{n+1}) - p(\psi_n)|}{r(\psi_n)}\right) \\
&= -GM\mu \lim_{N\to\infty}\left(\sum_{n=0}^{N-1} \frac{1}{r(\psi_n)}((p_1(\psi_{n+1}) - p_1(\psi_n))^2 + (p_2(\psi_{n+1}) - p_2(\psi_n))^2\right) \\
&= -GM\mu \lim_{N\to\infty}\left(\sum_{n=0}^{N-1} \frac{1}{r(\psi_n)} \sqrt{\dot p_1(\psi_{n1})^2 + \dot p_2(\psi_{n2})^2}\,(\psi_{n+1} - \psi_n)\right) \\
&= -GM\mu \int_{\varphi_1}^{\varphi_2} \frac{1}{r(\varphi)} \sqrt{r(\varphi)^2 + \dot r(\varphi)^2}\, d\varphi\,,
\end{aligned}$$

where $\psi_n < \psi_{n1}, \psi_{n2} < \psi_{n+1}$ by the mean value theorem.

More general, with a strictly increasing function $V :]0,\infty[\to \mathbb{R}$ we have to minimize the functional

$$F[r] := \int_{\varphi_1}^{\varphi_2} V(r(\varphi))\sqrt{r(\varphi)^2 + \dot{r}(\varphi)^2}\, d\varphi \tag{6}$$

among all sufficiently regular functions $r : [\varphi_1, \varphi_2] \to]0,\infty[$ with $r(\varphi_1) = r_1$ and $r(\varphi_2) = r_2$, satisfying the side condition

$$L[r] := \int_{\varphi_1}^{\varphi_2} \sqrt{r(\varphi)^2 + \dot{r}(\varphi)^2}\, d\varphi = 2l\,. \tag{7}$$

2 The solution

The *Euler–Lagrange equation* for the *isoperimetric* variational problem as given by (6) and (7) reads

$$\exists\lambda \in \mathbb{R} : f_r(r,\dot{r}) = \frac{d}{d\varphi} f_{\dot{r}}(r,\dot{r})\,, \tag{8}$$

where $f(r,\dot{r}) = (\lambda + V(r))\sqrt{r^2 + \dot{r}^2}$. It is a necessary condition for minimizers in C^2, a regularity property which will be established in the course of the existence proof in Section 5. Equation (8) is equivalent to

$$\frac{rV'(r)}{\sqrt{r^2 + \dot{r}^2}} = \kappa(\lambda + V(r))\,, \tag{9}$$

where κ is the curvature (i.e., derivative of the directional angle with respect to the arclength):

$$\kappa = \frac{r\ddot{r} - r^2 - 2\dot{r}^2}{(r^2 + \dot{r}^2)^{3/2}}\,, \tag{10}$$

where the sign has been chosen to be positive for the graphs of convex functions.

On the other hand, since f does not depend on φ explicitly, we have

$$\frac{d}{d\varphi}(f - \dot{r} f_{\dot{r}}) = \dot{r}(f_r - \dot{r} f_{r\dot{r}} - \ddot{r} f_{\dot{r}\dot{r}})\,,$$

such that we can immediately integrate the Euler–Lagrange equation, with the result that (8) is equivalent to

$$\exists\lambda \in \mathbb{R}\ \exists\alpha \in \mathbb{R} \setminus \{0\} : \frac{1}{\alpha} = (\lambda + V(r))\frac{r^2}{\sqrt{r^2 + \dot{r}^2}}\,, \tag{11}$$

except possibly for some artificially introduced constant solutions. We appear to have omitted the case $\alpha = \pm\infty$, where the constant of integration equals 0. Contrary to the first impression, this limiting case does not correspond to $V(r) + \lambda = 0$, r a constant. Indeed, by (9), (10) and $V' > 0$, we conclude from $\dot{r} \equiv 0$ that $V(r) + \lambda < 0$. We will see later that $|\alpha| \to \infty$ corresponds to $|\dot{r}| \to \infty$.

Since V is strictly increasing, (9) and (11) show that for any solution of (8) the curvature does not change sign. Those with positive curvature we will call *chains* ($\alpha > 0$) and those with negative curvature *arches* ($\alpha < 0$). The special case of the *taut chain* ("$\alpha = 0$") is not a solution of (8), since it corresponds to the case where there is only one admissible function, such that the method of Lagrange multipliers does not apply. It is no physical solution either, as the tension becomes infinite. It is possible to deduce some more information about the qualitative behavior of solutions, but we will now solve the problem explicitly for $V(r) = -\frac{1}{r}$ (it is clear that the values of G, M and μ do not affect potential minimizers of (6) as long as their product is positive), in which case (11) reduces to

$$\lambda r - 1 = \frac{1}{\alpha} \frac{\sqrt{r^2 + \dot{r}^2}}{r}. \tag{12}$$

It is obvious that the case $\lambda = 0$ is associated with $\alpha \leq -1$ and satisfies $r(\varphi) = \rho \exp(\pm\sqrt{\alpha^2 - 1}\varphi)$ with some $\rho > 0$; we will call it the *logarithmic case*, because the corresponding curve is a logarithmic spiral. Otherwise the substitution $\lambda r = \frac{1}{s}$, $\lambda \dot{r} = -\frac{\dot{s}}{s^2}$, transforms (12) into

$$\alpha(1 - s) = \operatorname{sign}(\lambda)\sqrt{s^2 + \dot{s}^2}. \tag{13}$$

Squaring and taking the derivative, we obtain $-\alpha^2(1 - s)\dot{s} = (s + \ddot{s})\dot{s}$. Cancelling $\dot{s}$, we get

$$\ddot{s} + (1 - \alpha^2)s = -\alpha^2 \tag{14}$$

on the complement of $\{\varphi \mid \dot{s}(\varphi) = 0\}$, i.e. everywhere, by continuity, since $\dot{s}$ cannot vanish on an open interval, because otherwise (10) would imply $r \equiv -1/\kappa$ on this interval, and then $\lambda = 0$ from (9). So we are left with solving (14), which is easy, but the solutions need to be checked by substitution into (13). Chains will always have a unique minimum φ_0 by convexity, so $\dot{s}(\varphi_0) = 0 = \dot{r}(\varphi_0)$. Therefore we can and will express the solutions for chains in terms of cos or cosh, without sin or sinh, viz

$$s(\varphi) = s_\alpha(\varphi - \varphi_0), \tag{15}$$

where the *cocatenary functions* s_α for $\alpha \neq -1$ are given by

$$s_\alpha(\psi) = \alpha \left(\frac{1}{\alpha + 1} - \sum_{n=1}^{\infty} \frac{(\alpha^2 - 1)^{n-1}}{(2n)!} \psi^{2n} \right) = \frac{\alpha(\alpha - \cosh(\sqrt{\alpha^2 - 1}\psi))}{\alpha^2 - 1}. \tag{16}$$

Note that we can include the arches ($\alpha < 0$), if we replace α by $-\alpha$ for the case where $\alpha \leq -1$ and $\lambda < 0$, which will be called the *hyperbolic case*, since it leads to a kind of hyperbolic spirals ($\alpha = -1$ and $\lambda > 0$ cannot occur together by virtue of (13)). We will, however, concentrate on the *standard case*, where $\alpha > -1$ or $\lambda > 0$, and postpone the justification and interpretation of the formula for $\alpha < 0$ until Section 3. We will then also explain the apparent discontinuities and ambiguities for the logarithmic and hyperbolic cases.

We now consider the *symmetric* boundary value problem of finding an appropriate chain or arch of given length $2l$ satisfying $r(-\Phi) = r_\Phi = r(\Phi)$. Any curve obeying these boundary conditions will have some point where $\dot{r} = 0$, and therefore by symmetry of s_α,

$\varphi_0 = 0$. Moreover the hyperbolic case does not occur, since then $r(0)$ would be negative; for $\lambda = 0$, α has to be -1 by (12) evaluated at $\varphi_0 = 0$.

Figure 5 shows Φ_α in dependence on α, where $]\varphi_0 - \Phi_\alpha, \varphi_0 + \Phi_\alpha[$ is the maximal symmetric interval where s is positive, i.e. the largest interval of existence of solutions to (12) with initial conditions $r(\varphi_0) = r_0$, $\dot r(\varphi_0) = 0$.

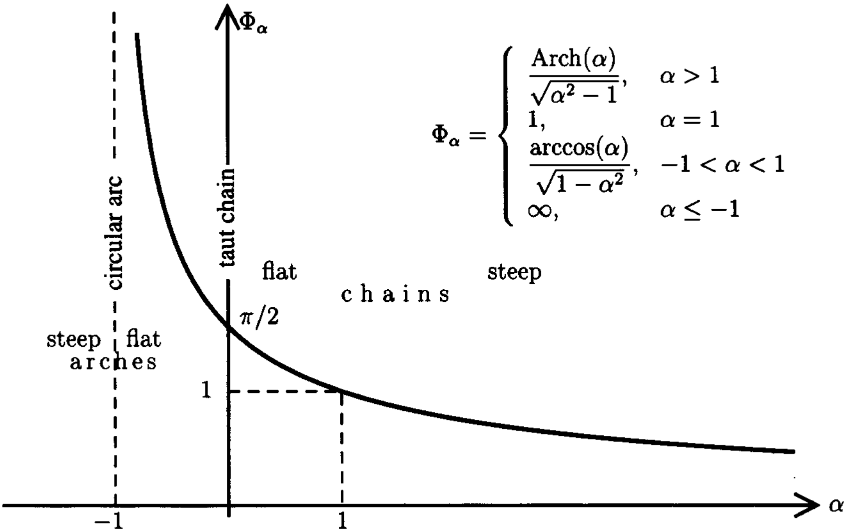


Figure 5: Length of the maximal domain of definition of extremals

To find the shape parameter α from the given length, we may disregard the case $\alpha = -1$ (where $r(\varphi) = r_\Phi$ and $l = \Phi r_\Phi$) and using (12) again, we get

$$\frac{d}{d\varphi}\sqrt{(\lambda r - 1)^2 - \alpha^{-2}} = \frac{\lambda(\lambda r - 1)\dot r}{\sqrt{(\lambda r - 1)^2 - \alpha^{-2}}} = \frac{\lambda(\lambda r - 1)\dot r}{\sqrt{\frac{r^2+\dot r^2}{\alpha^2 r^2} - \frac{1}{\alpha^2}}}$$
$$= \lambda(\lambda r - 1)\alpha r \,\mathrm{sign}(\alpha r/\dot r) = \lambda\sqrt{r^2 + \dot r^2}\,\mathrm{sign}(\alpha r/\dot r)$$

on $]0, \Phi]$.

Therefore,

$$l = \int_0^\Phi \sqrt{r(\varphi)^2 + \dot r(\varphi)^2}\,d\varphi = \frac{\mathrm{sign}(\alpha r/\dot r)}{\lambda}\left[\sqrt{(\lambda r - 1)^2 - \alpha^{-2}}\right]_{r=r_0}^{r=r_\Phi} = \frac{1}{\lambda}\frac{\dot r(\Phi)}{r_\Phi \alpha},$$

from which we get with $\eta := \sqrt{\alpha^2 - 1}\Phi$:

$$\frac{l}{\Phi r_\Phi} = \frac{\dot r(\Phi)}{\lambda\alpha\Phi r(\Phi)^2} = \frac{-\dot s(\Phi)}{\alpha\Phi} = \mathrm{sinch}(\eta)\,. \tag{17}$$

Here sinch is defined as in (3) on $i\mathbb{R}$ as well. By virtue of (5), the l.h.s. of (17) runs from $\frac{\sin(\Phi)}{\Phi}$ to $\frac{1}{\Phi}$, so that by the properties of the function sinch, (17) can uniquely be solved for $|\alpha|$. For arches, the upper limit on l is not necessary, and (17) can be solved uniquely for $\alpha < 0$ as the l.h.s. runs from $\frac{\sin(\Phi)}{\Phi}$ to ∞. Note that $\operatorname{sinch}(\eta) = 1$ for $\eta = 0$, i.e. for $|\alpha| = 1$, such that the circular arc $r(\varphi) = r_\Phi$ is again included.

The evaluation of α in terms according to (17) is depicted in Figure 6.

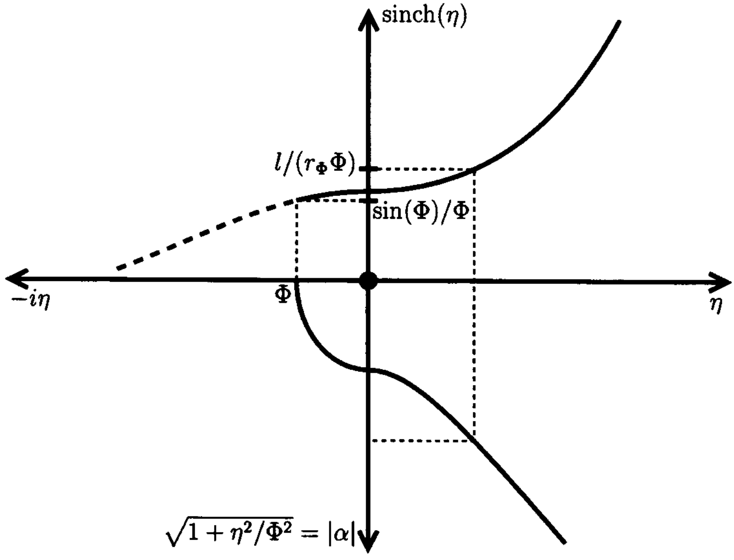


Figure 6: Determination of the shape parameter: given Φ, determine η (real or imaginary) from the upper curve, then determine $|\alpha|$ from the (Φ–dependent) lower curve

We have thus arrived at the following conclusion.

Theorem 1 *Let $\Phi \in \,]0, \frac{\pi}{2}[$ and $r_\Phi \in \,]0, \infty[$. For $l \in [r_\Phi \sin(\Phi), r_\Phi[$ the* Catenaria vera *of length $2l$ fixed in two points at a distance r_Φ from the origin with an angle of 2Φ is given by the function $r : [-\Phi, \Phi] \to \,]0, \infty[$ defined by*

$$\forall \varphi \in [-\Phi, \Phi] : r(\varphi) = r_\Phi \frac{s(\Phi)}{s(\varphi)}$$

with

$$s(\varphi) = \frac{\alpha - \cosh(\sqrt{\alpha^2 - 1}\varphi)}{\alpha - 1},$$

where $\alpha \geq 0$ is uniquely determined by

$$\frac{l}{\Phi r_\Phi} = \operatorname{sinch}(\sqrt{\alpha^2 - 1}\Phi).$$

If the last relation with $l \in]r_\Phi \sin(\Phi), \infty[$ *is solved for* $\alpha < 0$, *then* r *describes an upright arch of length* $2l$.

It is remarkable that the hyperbolic cosine actually reappears in the explicit formula for the true catenary!

In order to complete the proof of Theorem 1 it is necessary to show sufficiency, i.e. actual *minimality* of potential energy for the solutions of the Euler–Lagrange equation. We will approach this task by two different methods in Sections 5 and 6 respectively.

Another formidable task is to compare the true and classical catenaries for physical chains, since for all practical purposes and (to date) existing chains the difference is minute. For instance, the value $\Phi = \frac{1}{255}$ for a chain spanning 50km, i.e. of dimension $d \approx 25$km, hanging down ≈ 1km close to the Earth's surface ($r_\Phi \approx 6375$km), such that $\alpha \approx 20.4$, will lead to a difference between r_0 and $y(0)$, i.e. a difference in drop, of ≈ 2.7cm! However, huge chains can be simulated on a computer. The corresponding formulas for the parabola and the hyperbolic cosine are (with $d = r_\Phi \sin(\Phi)$):

$$\frac{y(x)}{d} = \cot(\Phi) - \frac{\gamma}{2}\left(1 - (\frac{x}{d})^2\right)$$

$$\frac{y(x)}{d} = \cot(\Phi) - \frac{1}{\beta}\left(\cosh(\beta) - \cosh(\beta\frac{x}{d})\right),$$

respectively (cf. (1) and (2)). In Figure 7 we represent the true catenary, the classical catenary and Galilei's parabola for the parameters $\Phi = 0.5$, $l = 1.5$, $d = 1$ (i.e. $r_\Phi \approx 2.086$).

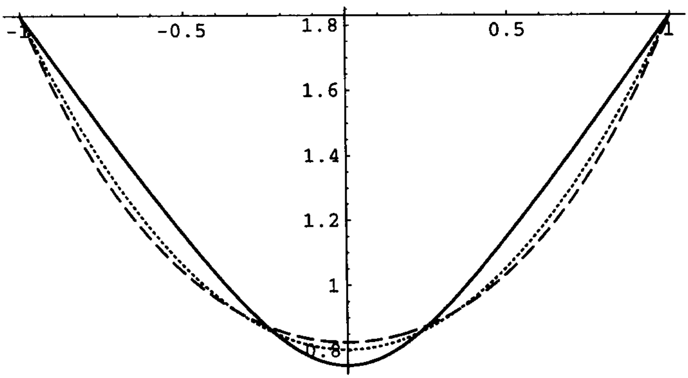


Figure 7: Galilei's parabola (dotted), the hyperbolic cosine (dashed), and the true catenary

A comparison of the numerical values for the potential energies, obtained from the formulas

$$-2\int_0^1 \sqrt{\frac{1+\gamma^2\xi^2}{\xi^2+[\cot(\Phi)-\frac{\gamma}{2}(1-\xi^2)]^2}}\,d\xi$$

for the parabola,

$$-2\int_0^1 \frac{\cosh(\beta\xi)}{\sqrt{\xi^2+[\cot(\Phi)-\frac{1}{\beta}(\cosh(\beta)-\cosh(\beta\xi))]^2}}\,d\xi,$$

for the hyperbolic cosine, and

$$-2\int_0^\Phi \left((\alpha+1)\frac{r(\varphi)}{r_0}-\alpha\right)d\varphi,$$

for the true catenary (cf. (13)), shows that in this case the parabola (with an energy value ≈ -2.4081) is closer to the true catenary (≈ -2.4295) than the hyperbolic cosine (≈ -2.3909)!

3 Asymmetric catenaries

In order to prove minimality of our solutions in Theorem 1 by means of the Weierstraß method, it will be essential to find unique solutions of the Euler–Lagrange equation for all admissible sets of data, so we will turn now to the full discussion of the solutions (15) for negative α. We have seen that chains automatically have a minimum φ_0. For arches with $\alpha \in\ ]-1,0[$, the differential equation (14) leads to trigonometric functions, so we can always choose *some* φ_0 such that $\dot{s}(\varphi_0)=0$ and get a cosine expression again. Its coefficient is determined from (13). The case $\alpha=-1$ in (14) is also straightforward; $\lambda<0$ is immediate from (13) in this case.

For $\alpha<-1$, (14) implies

$$s(\varphi)=\frac{\alpha^2}{\alpha^2-1}+A\cosh(\sqrt{\alpha^2-1}(\varphi-\varphi_0))+B\sinh(\sqrt{\alpha^2-1}(\varphi-\varphi_0))$$

for any arbitrarily prescribed φ_0, with constants of integration A, B. Writing (13) as $\operatorname{sign}(\lambda)\alpha(1-s)-s=\sqrt{s^2+\dot{s}^2}-s\geq 0$, we see that s is bounded either below or above, depending on the sign of λ, therefore $|B|\leq|A|$. For $|B|=|A|$, we get an exponential, and letting $\varphi\to\pm\infty$ such that $s\to 0$, $\dot{s}\to 0$, this implies $\lambda=0$. Therefore $|B|<|A|$, and we may assume $B=0$ at the price of choosing φ_0 appropriately. Exploiting (13) for $\varphi=\varphi_0$ yields

$$A=-\operatorname{sign}(\lambda)\frac{\alpha}{\alpha^2-1}\,.$$

This shows that (15) comprises the complete solution of (13). The meaning of φ_0 has been clear beforehand for $\alpha>0$, but for $\alpha<0$, the existence of some φ_0 with $\dot{s}(\varphi_0)=0$ has followed a posteriori. In this case, too, we let $r_0:=r(\varphi_0)$, and the relation between r_0 and λ is given by $r_0=\frac{\alpha+1}{\lambda\alpha}$ in the standard case and by $r_0=\frac{\alpha-1}{\lambda\alpha}$ in the hyperbolic case.

For $\alpha \in]-1,0[$ it follows from (13) that $\lambda < 0$, and we have an arch, but $r(\varphi)$ has a minimum, not a maximum, at φ_0. The arch is flatter than a circular arc. As α crosses 0 (from chains to arches through the taut chain), λ changes its sign from positive to negative by passing through ∞.

For the hyperbolic case, we have $r(\varphi_0) < 0$, so these represent spirals defined on a semi-infinite interval determined by the condition $r > 0$ ($s < 0$). There are two such intervals, one for an incoming, one for an outgoing spiral. The segment $r < 0$ ($s > 0$) between them can be interpreted as a chain at the antipodes by switching the sign of α. Finally, for $\alpha < -1$ and $\lambda > 0$, $r(\varphi_0)$ is a positive maximum, and we have steep arches.

We are now prepared to show that for any asymmetric boundary conditions $r(\varphi_1) = r_1 \neq r_2 = r(\varphi_2)$ ($0 < \varphi_2 - \varphi_1 =: 2\Phi < \pi$, cf. (4)) and any length $2l$ satisfying the geometric constraints (5), the upper bound being omitted for arches, there is a unique chain and a unique arch. To this end we assume w.l.o.g. that $\varphi_1 = 0$, so that the given data are $r_1, r_2, l \in]0,\infty[$, $\Phi \in]0, \frac{\pi}{2}[$ obeying (5); then $\varphi_2 = 2\Phi$. We try to determine one of our symmetric solutions, where the chain or arch with these data lies on, that is, we want to solve for $\alpha \in \mathbb{R}, \varphi_0 \in \mathbb{R}$ and $r_0 \in]0,\infty[$, such that

$$\forall\varphi \in]\varphi_0 - \Phi_\alpha, \varphi_0 + \Phi_\alpha[: \quad r(\varphi) = \frac{\alpha}{\alpha+1}\frac{r_0}{s_\alpha(\varphi - \varphi_0)}\,. \tag{18}$$

This will cover all but the logarithmic cases, provided we replace α by $-\alpha$ in the hyperbolic case (and account for a different domain then).

Since $r_1 = r(0)$ and $r_2 = r(2\Phi)$, and replacing φ_0 by $\psi_0 := \varphi_0 - \Phi$ for symmetry, (18) leads to (cf. (16)):

$$r_1\frac{\alpha - \cosh(\sqrt{\alpha^2-1}(\Phi+\psi_0))}{\alpha-1} - r_2\frac{\alpha - \cosh(\sqrt{\alpha^2-1}(\Phi-\psi_0))}{\alpha-1} = 0\,. \tag{19}$$

The length condition reads (cf. (17)):

$$r_1\frac{\sinh(\sqrt{\alpha^2-1}(\Phi+\psi_0))}{\sqrt{\alpha^2-1}} + r_2\frac{\sinh(\sqrt{\alpha^2-1}(\Phi-\psi_0))}{\sqrt{\alpha^2-1}} = 2l\,. \tag{20}$$

To solve for $\alpha, |\alpha| \neq 1$ first, we introduce the new variable $t := \tanh(\sqrt{\alpha^2-1}\frac{\psi_0}{2}) \in]-1,1[\cup i\mathbb{R}$, such that $\cosh(\sqrt{\alpha^2-1}\psi_0) = \frac{1+t^2}{1-t^2}$ and $\sinh(\sqrt{\alpha^2-1}\psi_0) = \frac{2t}{1-t^2}$. Note that the cases $t = \pm i\infty$ (for $|\alpha| < 1$) cannot occur, because $|\psi_0| < \Phi_\alpha < \pi/\sqrt{1-\alpha^2}$. If we insert this substitution into (19) and (20), we get

$$\begin{aligned} 0 &= (a+b)t^2 - 2ct + (b-a) \\ 0 &= (\tilde{a}+c)t^2 - 2bt + (c-\tilde{a}), \end{aligned} \tag{21}$$

where

$$a := (r_2 - r_1)\alpha \qquad b := (r_2 - r_1)\cosh(\eta) \qquad \text{and} \qquad \eta = \sqrt{\alpha^2-1}\Phi\,.$$
$$\tilde{a} := 2l\sqrt{\alpha^2-1} \qquad c := (r_2 + r_1)\sinh(\eta)$$

By Sylvester's elimination method, the equations (21) do have a common root if and only if their resultant vanishes (note that $\tilde{a} + c \neq 0$):

$$0 = \begin{vmatrix} a+b & -2c & b-a & 0 \\ 0 & a+b & -2c & b-a \\ \tilde{a}+c & -2b & c-\tilde{a} & 0 \\ 0 & \tilde{a}+c & -2b & c-\tilde{a} \end{vmatrix} .$$

This is equivalent to

$$0 = (c^2 - b^2)(c^2 - b^2 + a^2 - \tilde{a}^2),$$

whence $\tilde{a}^2 - a^2 = c^2 - b^2$ (the case $c^2 = b^2$, but $\tilde{a}^2 \neq a^2$ cannot occur, since then $t = \pm 1$, as can be seen by adding or subtracting the equations in (21)). This in turn amounts to

$$\left(\frac{l^2 - (\frac{r_2 - r_1}{2})^2}{\Phi^2 r_1 r_2} \right)^{1/2} = \operatorname{sinch}(\eta) \ . \tag{22}$$

As in (17), the l.h.s runs from $\frac{\sin(\Phi)}{\Phi}$ to $\frac{1}{\Phi}$ (or ∞ for arches), and we may use Figure 6 again to determine $|\alpha|$ uniquely from the given data.

Note that by the same procedure we can find the corresponding parameters β, x_0 and h of a classical catenary

$$\forall x \in [x_1, x_2] : \quad \frac{y(x)}{d} = \frac{h}{d} - \frac{1}{\beta} \left(\cosh(\beta) - \cosh(\beta \frac{x - x_0}{d}) \right) ,$$

if the suspension points are $(x_1, y_1), (x_2, y_2)$ and the dimension is defined by $d := \frac{x_2 - x_1}{2}$. In fact, the equations corresponding to (19) and (20) are

$$\beta \frac{y_2 - y_1}{d} = \cosh(\beta \frac{x_2 - x_0}{d}) - \cosh(\beta \frac{x_1 - x_0}{d})$$

$$\beta \frac{2l}{d} = \sinh(\beta \frac{x_2 - x_0}{d}) - \sinh(\beta \frac{x_1 - x_0}{d}) ,$$

and putting $t := \tanh(\frac{\beta x_0}{2d})$, $a := \beta \frac{y_2 - y_1}{d}$, $\tilde{a} := 2\beta \frac{l}{d}$, $b := \cosh(\beta \frac{x_2}{d}) - \cosh(\beta \frac{x_1}{d})$, $c := \sinh(\beta \frac{x_2}{d}) - \sinh(\beta \frac{x_1}{d})$, we arrive at (21) again and get β from

$$(\frac{l}{d})^2 - (\frac{y_2 - y_1}{2d})^2 = \operatorname{sinch}(\beta)^2$$

and then x_0 from

$$\frac{1}{\operatorname{sinch}(\beta)} \frac{y_2 - y_1}{2d} = \sinh \left(\beta (\frac{x_1 + x_2}{2} - x_0) \right) .$$

(Due to the simpler structure of the solutions, there is an easier way to get these parameters for the classical catenary; cf. e.g. [11, p. 373-375].) However, the situation is not as straightforward for the true catenary.

First of all, we left out the case $|\alpha| = 1$ in the derivation of (22), so that there will be a problem with the angle given by

$$\Phi = \Phi_h := \left(\frac{l^2 - (\frac{r_2 - r_1}{2})^2}{r_1 r_2} \right)^{1/2} , \tag{23}$$

which we now have to account for. For $\alpha = \pm 1$ the cocatenary function is given by $s_1(\varphi) = \frac{1}{2}(1-\varphi^2)$, so that (19) and (20) read

$$r_1(1-(\Phi+\psi_0)^2) = r_2(1-(\Phi-\psi_0)^2)\,,$$
$$(r_2-r_1)\psi_0 = (r_1+r_2)\Phi - 2l\,.$$

If $r_1 = r_2$, this leads to the solution $\psi_0 = 0$; otherwise,

$$\psi_0 = \frac{\frac{r_2+r_1}{2}\Phi - l}{\frac{r_2-r_1}{2}}\,.$$

The second problem which is considerably more involved than in the case of the classical catenary is to find φ_0 as soon as $\alpha \neq \pm 1$ is given, i.e. to guarantee that (21) has a unique solution $t \in \,]-1,1[\,\cup i\mathbb{R}$. We proceed by observing the sign of $A := c^2 - b^2 = \tilde{a}^2 - a^2$, which is 0 if and only if

$$|\alpha| = \frac{l}{\sqrt{l^2 - (\frac{r_2-r_1}{2})^2}}\,.$$

The cases $b = \pm c$ are equivalent to $t = \pm 1$ being a solution of (21) (signs never to be read crosswise); furthermore, $|\alpha| > 1$. The second solution of the second equation in (21) is then $t = \frac{b\mp\tilde{a}}{c+\tilde{a}}$, which lies in $]-1,1[$, if and only if $\mathrm{sign}(r_2 - r_1) = \pm 1$, and is a solution of the first equation in (21), if and only if $a = \pm\tilde{a}$, i.e. $\alpha > 1$. This is also the only case when two simultaneous solutions of (21) exist. Otherwise t is unique, and we only have to make sure that it lies in $]-1,1[$ if $|\alpha| > 1$. Since $t = \frac{\tilde{a}b - ac}{B}$, where $B := A + \tilde{a}c - ab$, we have $|t| < 1 \Leftrightarrow AB > 0$. If $A > 0$, then $ab < \tilde{a}c$, whence $B > A > 0$ as well. Otherwise, since $(\tilde{a}c - ab)^2 \geq A^2$ and $|ab| > \tilde{a}c$, we have to make sure that $0 < ab = \alpha(r_2-r_1)^2\cosh(\eta)$, i.e. $\alpha > 0$. But this is obvious in the standard case and was forced in the hyperbolic case.

We are left with the logarithmic case where $\lambda = 0$, i.e. $r(\varphi) = r_1\exp(\pm\sqrt{\alpha^2-1}\varphi)$ and $\alpha < -1$. In particular, $r_2 = r_1\exp(\pm 2\eta)$, which is equivalent to $b = \pm c$, and the length condition yields

$$2l = \frac{r_1|\alpha|}{\pm\sqrt{\alpha^2-1}}(e^{\pm 2\eta} - 1) = \frac{(r_2-r_1)(-\alpha)}{\pm\sqrt{\alpha^2-1}}\,,$$

which is the same as $a = \mp\tilde{a}$. So this is just the case missing in the above discussion, namely

$$\alpha = \alpha_l := -\frac{l}{\sqrt{l^2 - (\frac{r_2-r_1}{2})^2}}\,, \quad \Phi = \Phi_l := \frac{\ln(r_2) - \ln(r_1)}{r_2 - r_1}\sqrt{l^2 - (\frac{r_2-r_1}{2})^2}\,.$$

Note that this angle is less than the Φ_h for the hyperbolic spiral given by (23) and that they both tend to $\frac{l}{r_\Phi}$ for the symmetric case $r_1 = r_\Phi = r_2$, where $\alpha = -1$ and the solutions merge into the circular arc.

We summarize the results of this section in the following theorem.

Theorem 2 *Let $r_1, r_2 \in]0, \infty[$, $l \in]\frac{r_2-r_1}{2}, \frac{r_2+r_1}{2}[$; then Φ_h from (23) satisfies $\Phi_h \in]0,1[$. If we let the parameter α run from $-\infty$ to ∞, we encounter the following cases for the curve r of length $2l$ spanned between the points given by $r(0) = r_1$ and $r(2\Phi) = r_2$, where Φ is uniquely determined by*

$$\operatorname{sinch}(\sqrt{\alpha^2 - 1}\Phi) = \frac{\Phi_h}{\Phi} :$$

for $-\infty < \alpha < \alpha_l$, Φ runs from 0 to Φ_l and we have a steep arch as given by (18);

for $\alpha = \alpha_l$ and $\Phi = \Phi_l$ we have the logarithmic spiral

$$r(\varphi) = r_1 \exp\left(\pm\sqrt{\alpha_l^2 - 1}\,\varphi\right);$$

for $\alpha_l < \alpha \leq -1$, Φ runs from Φ_l to and including Φ_h and we have an arch on a hyperbolic spiral as given by (18), but with α replaced by $-\alpha$;

for $-1 < \alpha < 0$, Φ runs from Φ_h to $\arcsin(\Phi_h)$ *and we have a flat arch as given by (18);*

for $\alpha = 0, \Phi =$ $\arcsin(\Phi_h)$ *we have the taut chain;*

for $0 < \alpha < \infty$, Φ runs back from $\arcsin(\Phi_h)$ *to 0 and we have true chains as given by (18).*

(Note that $\alpha = -\infty$ and $\alpha = \infty$ appear as borderline cases with $\Phi = 0$ and represent the Indian rope trick *and the* slack rope, *respectively.)*

4 An overview of methods for an existence proof

The proof that the classical catenary actually minimizes the corresponding functional is hardly found in textbooks at all, even in those that do discuss both the catenary and the tools for establishing minimality. An honorable exception is the rather recent textbook by Troutman [14].

There are several strategies to establish that a solution of the Euler–Lagrange equations is actually a minimum of the functional under discussion. We will examine them in turn.

4.1 Second variation

The most popular method among introductory calculus students (or rather their teachers) is to examine the second derivative. In the calculus of variations setting, this approach (via the second variation) poses some technical and conceptual difficulties and leads to the discussion of *conjugate points.* By its very nature, a second derivative test can only show that some given candidate is a relative (local) minimizer, as the test uses only local information. It should be pointed out that even in the case of functions of two variables, a relative minimum that happens to be the only critical point need not be an absolute minimum [12] (due to the possible presence of "saddle points at infinity").

For the catenary, which has to be shown to be a minimizer under constraints, the second variation approach is less feasible.

The method of fields of extremals is an enhancement of the method of the second variation, which, if applicable, can also show a candidate to be an absolute minimizer. A predecessor of the method, due to Weierstraß, can also be applied in the case of constraints (i.e., the case with Lagrange multipliers). This has been done by Kneser [9] for the classical catenary. Alas, this paper doesn't seem to be well known, and we are grateful to H. Kalf for pointing it out to us. This method will be exploited for the case of the catenaria vera below in Section 6.

All methods referred to in this section were available in the 19th century and have been developed mainly by Weierstraß, Jacobi, and Hilbert.

4.2 Direct methods

A modern approach is the one by direct methods: a lower semi-continuous function on a compact set takes on a minimum. The tools to exploit this principle were not available to the founders of the calculus of variations. In practice, one has to work, say, with Sobolev spaces and the weak topology therein in order to get a compromise between the compactness requirement (which needs a sufficiently weak topology) and the semi-continuity requirement (which needs a sufficiently strong topology). These methods have been worked out in a pre-Sobolev space version (which admits even more general assumptions) by Tonelli [13]. In any case, Lebesgue's theory of integration is required.

The particular nature of the functional for the catenary, namely the integrand's failure to have superlinear growth in the first derivative of the unknown function, makes it a borderline case, where direct methods encounter difficulties. Nevertheless we will show how an existence proof by direct methods works in this case. See Section 5 below.

By nature, direct methods show the existence of absolute minimizers. They do so in a wider class of functions than what would be admissible for classical (C^1) solutions of the variational problem, let alone classically admissible for the Euler–Lagrange equations (C^2). Unlike these latter, direct methods only depend on continuity, but not on differentiability properties of the functional. Therefore one may have to face the following inconveniences:

- Minimizers need not satisfy the Euler–Lagrange equations in any weak sense. (The functional may lack differentiability, even if an attempt to evaluate $\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} F[u+\varepsilon\varphi]$ formally for nice variations φ may not exhibit any trouble.)
- Minimizers need not be smooth functions.
- The minimum over the broader class of functions possessing just as much regularity for the functional to be defined may be strictly less than the infimum over smooth

functions (*Lavrentiev phenomenon*).

All these things can actually turn up for quite unconspicuous functionals [1, 5], but in classical examples with a physical background they can often be shown not to occur.

We will show below that the minimizer found by direct methods in our case is actually a smooth function and coincides with the catenaria vera found as a candidate from the Euler–Lagrange equations.

4.3 Convexity arguments

The method via convexity is less versatile in its range of applicability, but is actually the most elementary one. If a functional is convex, then any critical point is automatically an absolute minimum, and uniquely so in the case of strict convexity. This method, if applicable, can be appended to the solution of the Euler–Lagrange equation without much ado and replace the discussion of the second variation and fields of extremals, occasionally even in the presence of Lagrange multipliers. This approach has been stressed by Troutman, who handles the classical catenary that way [14].

The convexity argument is related to direct methods by the fact that convexity of the functional in the highest derivative is sufficient for lower semi-continuity, and if a critical point is found, convexity of the functional also guarantees regularity of the minimizer.

In the very short Section 7, we will see that Troutman's approach does not generalize to central potentials.

5 Direct methods

In this section, we show the existence of a minimizer for the functional F given in (6) among all functions $\varphi \mapsto r(\varphi)$ lying in the Sobolev space $W^{1,1}([\varphi_1, \varphi_2])$, i.e., the space of absolutely continuous functions, and satisfying the boundary conditions $r(\varphi_1) = r_1$, $r(\varphi_2) = r_2$ and the length constraint (7). Then we will reason that such a minimizer satisfies the Euler–Lagrange equation in some weak sense, and that this implies that the minimizer is actually smooth and satisfies the Euler–Lagrange equation in the classical sense. We will not discuss maximizers, and as arches cannot be minimizers (cf. the third picture in Figure 3), we may restrict the discussion to chains.

Compared to an existence proof in the space $W^{1,2}$ consisting of those absolutely continuous functions whose derivatives are square integrable, an existence proof in $W^{1,1}$ is less straightforward due to functional analytic complications. The logical scheme to overcome this difficulty is as follows:

1. An a-priori estimate: *If* there exists a minimizer in $W^{1,1}$, then it is actually smooth, satisfies the Euler–Lagrange equation, and its derivative can be bounded above in terms of the boundary data and the length constraint.

2. Therefore, any v that is not Lipschitz cannot be a minimizer, i.e., there must exist some $w \in W^{1,1}$ and some $\varepsilon > 0$ such that $F[w] \le F[v] - 2\varepsilon$. Using the facts that Lipschitz functions are dense in $W^{1,1}$ and F is continuous with respect to the $W^{1,1}$ topology, we find a Lipschitz function $\tilde{w}$ such that $F[\tilde{w}] \le F[v] - \varepsilon$.

3. We consider a sequence of functionals F_N such that for $N_1 < N_2$ it holds $F_{N_1} \ge F_{N_2} \ge F$ and such that F_N agrees with F on the set of Lipschitz functions with Lipschitz constant N.

   Unlike F, F_N will have the property that $F_N \equiv +\infty$ on $W^{1,1} \setminus W^{1,2}$, but $F_N < \infty$ on $W^{1,2}$. So we will carry out a straightforward existence proof of a minimizer r_N for F_N in the space $W^{1,2}$. And more or less the same argument as in the a priori estimate will show that r_N is Lipschitz with some Lipschitz constant that remains bounded as $N \to \infty$. We may extract a subsequence from r_N that converges to some r.

4. Going to the limit in $F[r_N] = F_N[r_N] \le F_N[v]$ $(\forall v \in W^{1,1})$ yields that for any v, $F[r] \le F[v]$, or else $F_N[v] = \infty$ for all N. However, the latter case implies v is not Lipschitz, and therefore there is some Lipschitz function $\tilde{w}$ satisfying $F[\tilde{w}] < F[v]$. But $\tilde{w}$ belongs to the former case, and therefore $F[r] \le F[\tilde{w}] < F[v]$ again.

Let us now do the details. For the potential, we assume $V \in C^1(]0,\infty[)$ and $V' > 0$.

Supposing $r \in W^{1,1}$ minimizes F within the class given by $L[r] = 2l$, our choice of l (longer than the straight line) guarantees that r is not a critical point of L, so we choose two arbitrary Lipschitz functions v_1, v_2 with zero boundary values, only subject to the condition $0 \ne DL[r]v_2 = \int (rv_2 + \dot{r}\dot{v}_2)/\sqrt{r^2 + \dot{r}^2}$, and we consider the family $r + \varepsilon_1 v_1 + \varepsilon_2 v_2$, upon which the length constraint $L = 2l$ imposes a condition $\varepsilon_2 = \varepsilon_2(\varepsilon_1)$ by the implicit function theorem in $\mathbb{R}^2$. It is then straitghforward to calculate

$$0 = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} F[r + \varepsilon v_1 + \varepsilon_2(\varepsilon) v_2] = DF[r]v_1 + \lambda DL[r]v_1,$$

where $\lambda = -DF[r]v_2/DL[r]v_2$. In this argument, $DF[r]v$ and $DL[r]v$ need to be defined only as directional derivatives in certain directions $v \in W^{1,1}$, namely in the directions of Lipschitz functions; in this sense they actually are defined. This gives the weak Euler–Lagrange equation

$$\int (\partial_1 f(r,\dot{r})v + \partial_2 f(r,\dot{r})\dot{v})\, d\varphi = 0 \quad (\forall v \in W_0^{1,\infty})$$

without any extra regularity assumption; here, $f(r,\dot{r}) = (\lambda + V(r))\sqrt{r^2 + \dot{r}^2}$. Integrating the term with v (not the one with $\dot{v}$ as is often done) by parts and using the fundamental lemma (see, e.g., Chapter 9 of [3]) produces

$$\frac{\dot{r}(V(r)+\lambda)}{\sqrt{r^2+\dot{r}^2}} - \int^{\varphi} \left(V'(r)\sqrt{r^2+\dot{r}^2} + \frac{r(V(r)+\lambda)}{\sqrt{r^2+\dot{r}^2}} \right) d\varphi \equiv const.$$

Therefore $\frac{\dot{r}}{\sqrt{r^2+\dot{r}^2}}$ and hence $\dot{r}$ is continuous, wherever $V(r) + \lambda \ne 0$. Note that it is the strict convexity of f with respect to the $\dot{r}$ variable that allows the conclusion $\dot{r} \in C^0$. This argument can be repeated ro give $\dot r \in C^1$, and then we get the classical Euler equations (9)–(11) in any interval where $V(r)+\lambda \neq 0$. By (11), this latter condition prevails on the closure of such an interval as well, so it holds either everywhere or nowhere. As we have seen in Section 2, right after (11), $V(r)+\lambda$ cannot vanish on any open interval, so we get the Euler equations everywhere.

Now we show the a priori estimates. There exist constants $a > 0$ and $b > 0$ such that any curve obeying the constraints will satisfy $a \le r \le b$. Moreover, there exists (very trivially) a constant $c > 0$ such that $|\varphi_2 - \varphi_1| \ge c$. The a priori estimates to be deduced will only depend on (a, b, c). If the lowest point (φ_0, r_0) of the extremal lies on the physical chain segment, we also have $r_0 \ge a$. Otherwise that point lies above the straight line joining the endpoints of the chain, and we still get the lower bound

$$r_0 > \frac{r_1 r_2 \sin(\varphi_2 - \varphi_1)}{(r_1^2 + r_2^2 - 2r_1 r_2 \cos(\varphi_2 - \varphi_1))^{1/2}} > \frac{a^2 \sin(c)}{2b} .$$

Now use (9) and (10) in connection with (11) to obtain the Euler–Lagrange equation in the form

$$\ddot r = \frac{V'(r)(r^2 + \dot r^2)}{V(r) - V(r_0) + 1/(\alpha r_0)} + \frac{2\dot r^2}{r} + r \ge \frac{c_0}{c_1(r - r_0) + 1/(\alpha r_0)} .$$

Integrating yields

$$\dot r^2 \ge \frac{2c_0}{c_1} \log\left(1 + \alpha r_0 c_1 (r - r_0)\right),$$

whence

$$\begin{aligned} \sqrt{\frac{2c_0}{c_1}}|\varphi_2 - \varphi_0| &\le \int_{r_0}^{r_2} \frac{dr}{\sqrt{\log\left(1 + \alpha r_0 c_1 (r - r_0)\right)}} \\ &\le \left[\log\left(1 + \alpha r_0 c_1 (r_2 - r_0)\right)\right]^{-1/2} \int_{r_0}^{r_2} \sqrt{\frac{r_2 - r_0}{r - r_0}}\, dr . \end{aligned}$$

Therefore, if $\alpha \to \infty$, we would conclude $\varphi_2 - \varphi_0 \to 0$, and similarly for φ_1. But $|\varphi_2 - \varphi_1| \ge c$, and this gives an upper bound for α depending on a, b, c only. Now by (11), $\lambda + V(r)$ is bounded away from 0, and then by (9), κ is bounded above. On the other hand, we get an upper bound for λ, because $\lambda \to \infty$ would imply $\kappa \to 0$ uniformly on the whole chain segment by (9), so the chain would come close to a taut chain, for which $|\dot r| \le C(a, b, c)$. So either we have, say $|\dot r| \le 2C(a, b, c)$, or else we get a bound for λ in terms of a, b, c. In this latter case, the Euler–Lagrange equation in the form

$$\ddot r = \frac{V'(r)(r^2 + \dot r^2)}{V(r) + \lambda} + \frac{2\dot r^2}{r} + r \le \left(\frac{V'(r)}{V(r) + \lambda} + \frac{2}{r}\right) \alpha^2 r^4 (V(r) + \lambda)^2$$

gives an upper bound for $\ddot r$ and hence for $\dot r$. In either case, we have shown $\dot r \le K(a, b, c)$ for some function K.

Existence of a minimizer for the modified functional $F_N[r] := F[r] + \int W_N(\dot r)\, d\varphi$, where $W_N \equiv 0$ on $[-N, N]$, $W_N(\dot r) = (|\dot r| - N)^2$ for $|\dot r| > N + 1$, and W_N convex, C^2

in between, is straightforward. The functional controls the $W^{1,2}$ norm, and a minimizing sequence contains a weakly convergent subsequence, because $W^{1,2}$ is reflexive. The functional is lower semi-continuous with respect to the weak topology. (As the modified functional is no longer bounded above on $W^{1,1}$, a different approach would be needed for maximizers.) The argument that minimizers of the modified problem satisfy the classical Euler–Lagrange equations works just as before. We get

$$\frac{rV'(r)}{\sqrt{r^2+\dot{r}^2}} = \kappa(\lambda + V(r)) + \frac{W_N''(\dot{r})\ddot{r}}{r} \tag{24}$$

and

$$\frac{1}{\alpha} = (\lambda + V(r))\frac{r^2}{\sqrt{r^2+\dot{r}^2}} + (W_N(\dot{r}) - \dot{r}W_N'(\dot{r})).$$

We employ this existence result for $N > N_0$ only. Here, N_0 is chosen large enough. Firstly we want $N_0 > K(a, b, c/2)$. Moreover, we assume $N_0 > |r_2 - r_1|/(\varphi_2 - \varphi_1)$, such that any C^1 curve connecting the endpoints must satisfy $|\dot{r}| < N_0$ on some open set, as a consequence of the mean value theorem. We will call some arbitrary maximal subinterval of the set $|\dot{r}| < N$ a *core* of the segment. On a core, the unmodified equations hold, and we will also have $\kappa > 0$, $\alpha > 0$, and $V(r) + \lambda > 0$ there. Finally, we want $N_0 > 2(r_2 + r_1)/(\varphi_2 - \varphi_1)$.

Extending the minimizing extremal for F_N to its maximal domain of definition as a solution to an ODE, we claim that $\ddot{r} > 0$ everywhere. This is certainly true for a core, which is delimited by points with $\dot{r} = \pm N$, where even $\kappa > 0$ holds (cf. (10)). On a maximal interval containing the core and on which $\ddot{r} \geq 0$, we will still have $\lambda + V(r) > 0$ because of the monotonicity of V; in a boundary point of that interval, we would have $\ddot{r} = 0$, and consequently $\kappa < 0$, such that the right hand side of (24) would be negative, whereas the left hand side is positive. Therefore no such boundary point exists, and $\ddot{r} > 0$ everywhere. In particular, our extremal for F_N consists of a (single) core, on which it coincides with an extremal for F, and (possibly) of adjacent intervals where the extremal satisfies $\dot{r} < -N$ and $\dot{r} > N$, respectively. If the complement of the core within the full extremal intersects $[\varphi_1, \varphi_2]$ at all, it can do so in at most two intervals of compound length $\frac{r_1}{N} + \frac{r_2}{N}$, because $|\dot{r}| > N$ outside the core. By virtue of $N > N_0 > 2(r_2 + r_1)/(\varphi_2 - \varphi_1)$, at least half of the segment consists of the core part, and on this core part within the segment, the a priori estimate guarantees $|\dot{r}| \leq K(a, b, c/2) < N_0$. As $|\dot{r}| > N > N_0$ outside the core, the points delimiting the core from the rest must lie outside the segment. For this segment, r_N, we therefore have

$$F[r_N] = F_N[r_N] \leq F_N[\tilde{r}] \quad (\forall \tilde{r})$$

and letting $N \to \infty$ shows that $r_N \to: r$ (possibly on a subsequence) where r is a minimizer for F. Of course, if minimizers are unique, as one may reasonably suspect in this case, (r_N) is an eventually constant sequence.

Having shown the existence of minimizers and their automatic regularity, we are now sure that the unique solutions of the necessary conditions, which were found in Sections 2 and 3 under the assumption of existence and regularity, are indeed minimizers.

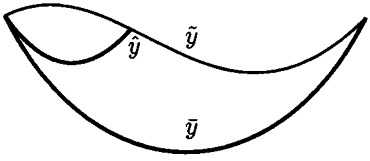
6 Minimality proof for an extremal by Weierstraß's method

Usually, an extremal (i.e., a segment satisfying the Euler–Lagrange equations) is shown to be a minimizer by embedding it into a field of extremals and discussing Weierstraß's $\mathfrak{E}$-function:

For y to minimize $F[y] := \int f(t, y(t), \dot{y}(t))\,dt$ (in the absence of constraints), it is necessary that $f_{\dot{y}\dot{y}} \geq 0$ along the extremal (*Legendre's condition*). This is the second derivative test applied (only) to variations supported on very short intervals. On the other hand, two minimal segments cannot intersect twice in their interior, because otherwise a minimal segment with corners could be patched together from them, whereas $f_{\dot{y}\dot{y}} > 0$ prevents extremals with corners. The essence of *Jacobi's condition*, dealing with C^1-small variations (whose support may now be a large interval), is therefore to exclude the existence of neighbouring extremals which intersect twice. Variations that are only C^0-small, but not necessarily C^1-small, are dealt with by *Weierstraß's condition*. All this is lucidly explained in Chapter 5 of [3] and leads naturally to sufficient conditions for minimality with respect to the various types of variations. However, its generalization to the case with isoperimetric constraints is not so clear, and it is hardly discussed in textbooks. In [4], Sections 453–457, this question is discussed as a special case of the more difficult situation of *Lagrangian constraints* (Sections 421–452), where the functions must satisfy pointwise conditions rather than integral conditions.

We avoid these intricacies by resurrecting a less general predecessor of the theory of extremal fields, due to Weierstraß. For the classical catenary, the argument has been carried out by Kneser [9]. It relies on our ability to construct a *unique* extremal segment for arbitrary boundary and length data (subject to the obvious restrictions). We follow Kneser, but in a modernized language.

In order to show that an extremal $t \mapsto \bar{y}(t)$, $t \in [a, b]$ is actually a minimizer of the functional $y \mapsto F[y] := \int_a^b f(t, y(t), \dot{y}(t))\,dt$ we proceed as follows: given an arbitrary comparison curve $\tilde{y}$ satisfying the same boundary conditions as $\bar{y}$, construct a homotopy $\tau \mapsto \hat{y}_\tau := \hat{y}(\cdot, \tau)$ from $\tilde{y}$ to $\bar{y}$ and show $\frac{d}{d\tau} F[\hat{y}_\tau] \leq 0$. To this end, let $\hat{y}_\tau$ coincide with $\tilde{y}$ on the interval $[\tau, b]$ and be extremal on the interval $[a, \tau]$. We will have the equations shown in Figure 8.



$$\begin{aligned}
&\hat{y}(\tau, \tau) = \tilde{y}(\tau) \\
&\hat{y}(a, \tau) = \tilde{y}(a) = \bar{y}(a) \\
&\tilde{y}(b) = \bar{y}(b) \\
&\tfrac{\partial}{\partial t}\hat{y}(t, \tau)|_{t=\tau+} = \dot{\tilde{y}}(\tau) \\
&\tfrac{\partial}{\partial t}\hat{y}(t, \tau)|_{t=\tau-} =: \hat{p}(\tau)
\end{aligned}$$

Figure 8: Scheme of Weierstraß's proof

Then a calculation given below yields

$$\frac{d}{d\tau}F[\hat{y}_\tau] = -\mathfrak{E}(\tau, \tilde{y}(\tau), \hat{p}(\tau), \dot{\tilde{y}}(\tau)), \tag{25}$$

where

$$\mathfrak{E}(t, y, \dot{y}, p) := f(t, y, p) - f(t, y, \dot{y}) - f_{\dot{y}}(t, y, \dot{y})(p - \dot{y}).$$

It is known (cf. [3, Chapter 5]) that $\mathfrak{E}(t, \bar{y}(t), \dot{\bar{y}}(t), p) \geq 0$ for all t, p is a necessary condition for $\bar{y}$ to be a minimizer with respect to all competitors within a C^0 neighbourhood. If the above construction is possible, then it shows that a similar condition is also sufficient, i.e., $F[\bar{y}] \leq F[\tilde{y}]$, provided $\mathfrak{E}(t, \tilde{y}, \dot{\tilde{y}}, p) \geq 0$ for all t, p and for all pairs $(\tilde{y}, \dot{\tilde{y}})$ that are actually taken on for some t by the competitor $\tilde{y}(\cdot)$. However, the construction is likely to fail whenever some points may be joined by different extremals. Then an attempt to carry out the above construction may end up with a $\hat{y}(\cdot, b) \neq \bar{y}(\cdot)$.

Indeed, let us check equation (25).

$$\begin{aligned}\frac{d}{d\tau}F[\hat{y}_\tau] &= \frac{d}{d\tau}\int_a^\tau f(t, \hat{y}_\tau(t), \dot{\hat{y}}_\tau(t))\,dt + \frac{d}{d\tau}\int_\tau^b f(t, \tilde{y}(t), \dot{\tilde{y}}(t))\,dt \\ &= f(\tau, \hat{y}_\tau(\tau), \dot{\hat{y}}_\tau(\tau-)) - f(\tau, \tilde{y}(\tau), \dot{\tilde{y}}(\tau)) + \int_a^\tau \frac{\partial}{\partial\tau} f(t, \hat{y}_\tau(t), \dot{\hat{y}}_\tau(t))\,dt,\end{aligned}$$

where

$$\begin{aligned}&\frac{\partial}{\partial\tau} f(t, \hat{y}_\tau(t), \dot{\hat{y}}_\tau(t)) = \\ &\quad = \left(f_y - \frac{\partial}{\partial t}f_p\right)\left(t, \hat{y}_\tau(t), \dot{\hat{y}}_\tau(t)\right)\frac{\partial}{\partial\tau}\hat{y}_\tau(t) + \frac{\partial}{\partial t}\left(f_p\left(t, \hat{y}_\tau(t), \dot{\hat{y}}_\tau(t)\right)\frac{\partial}{\partial\tau}\hat{y}_\tau(t)\right).\end{aligned}$$

The first term vanishes due to the Euler–Lagrange equations, and the second term can be integrated immediately. We get $\frac{\partial}{\partial\tau}\hat{y}_\tau(t)|_{t=\tau-} + \frac{\partial}{\partial t}\hat{y}_\tau(t)|_{t=\tau-} = \dot{\tilde{y}}(\tau)$ from differentiating $\hat{y}_\tau(\tau) = \tilde{y}(\tau)$.

The very same approach applies when discussing minimization problems for $F[y]$ under the constraint $L[y] = \int_a^b \ell(t, y(t), \dot{y}(t))\,dt = 2l = \text{const}$. Then $\tilde{y}$ will be assumed to obey the constraint, and so will $\hat{y}_\tau$ for all τ. For each τ, $\hat{y}|_{[a,\tau]}$ will satisfy the Euler–Lagrange equations for $f + \lambda(\tau)\ell$, but the Lagrange multiplier λ may depend on τ.

Evaluating $0 = \frac{d}{d\tau}L[\hat{y}_\tau]$ in the same way as $\frac{d}{d\tau}F[\hat{y}_\tau]$, we now get

$$\frac{d}{d\tau}F[\hat{y}_\tau] = -\mathfrak{E}_f(\tau, \tilde{y}(\tau), \hat{p}(\tau), \dot{\tilde{y}}(\tau)) + \int_a^\tau \left(f_y - \frac{\partial}{\partial t}f_p\right)\left(t, \hat{y}_\tau(t), \dot{\hat{y}}_\tau(t)\right)\frac{\partial}{\partial\tau}\hat{y}_\tau(t)\,dt$$

and

$$0 = \frac{d}{d\tau}L[\hat{y}_\tau] = -\mathfrak{E}_\ell(\tau, \tilde{y}(\tau), \hat{p}(\tau), \dot{\tilde{y}}(\tau)) + \int_a^\tau \left(\ell_y - \frac{\partial}{\partial t}\ell_p\right)\left(t, \hat{y}_\tau(t), \dot{\hat{y}}_\tau(t)\right)\frac{\partial}{\partial\tau}\hat{y}_\tau(t)\,dt$$

and therefore, using the appropriate Lagrange multiplier $\lambda(\tau)$,

$$\frac{d}{d\tau}F[\hat{y}_\tau] = -\mathfrak{E}_{f+\lambda(\tau)\ell}(\tau, \tilde{y}(\tau), \hat{p}(\tau), \dot{\tilde{y}}(\tau)).$$

We have seen in Section 3 that for all boundary and length data satisfying the obvious geometric constraints, there is a unique chain connecting them. In the presence of our a priori estimates (Section 5), uniqueness implies continuous dependence on τ: if $\tau_n \to \tau$, the corresponding y_n must have a point of accumulation, and each such point of accumulation is a (i.e., the unique) solution for τ. For the $-1/r$-potential, our explicit calculations show that the parameters $(r_0, \varphi_0, \lambda)$ depend smoothly on the data $(\varphi_1, \varphi_2, r_1, r_2, l)$, so $\tau \mapsto y_\tau$ is C^1 $(W^{1,1})$, if $\tilde{y}$ is. Therefore the construction in Figure 8 can be carried out, unless the comparison candidate $\tilde{y}$ starts with a straight line segment. This exception can be avoided by prolongating both $\bar{y}$ and $\tilde{y}$ to the left with a piece of the extremal of which $\bar{y}$ is a part.

For the given functionals we have

$$\mathfrak{E}_{f+\lambda\ell}(\varphi, r, p, \dot{r}) = (V(r) + \lambda)\left(\sqrt{r^2+\dot{r}^2} - \sqrt{r^2+p^2} - (\dot{r}-p)\frac{p}{\sqrt{r^2+p^2}}\right).$$

The second factor is positive because $p \mapsto \sqrt{r^2+p^2}$ is strictly convex for any value of r, and we have seen in (9) that $V(r) + \lambda$ is positive if r describes a chain. This completes the sufficiency proof. The argument works for arbitrary central potentials, provided the uniqueness part (and smooth dependence) can be carried over.

The argument could have been formulated in such a way that the comparison curve may be a space curve [9], so this proof supersedes most of the reduction process in Figure 3.

7 An attempt at joint convexity in $(r, \dot{r})$

Whereas arguments involving the Weierstraß $\mathfrak{E}$-function typically exploit the convexity of the integrand $f(t, y, \dot{y})$ with respect to the last variable $\dot{y}$, a simpler argument is available in the much rarer situation of convexity with respect to both variables $(y, \dot{y})$. This strategy has been employed by Troutman to give a minimality proof for the classical catenary by considering the vertical coordinate as a function of the arclength rather than as a function of the horizontal coordinate. However, the corresponding approach does not work for any of the more important rotationally symmetric potentials. (Of course, this does not exclude the possibility that fancier coordinates could do the trick.) We now document the failure of the method.

The general principle is as follows: if we have some solution y_0 to the Euler–Lagrange equation corresponding to $\tilde{f} := f + \lambda\ell$, and if $y \mapsto (F + \lambda L)[y]$ is convex (for which it is sufficient that $(y, p) \mapsto (f + \lambda\ell)(t, y, p)$ is convex for each t), then y_0 minimizes $(F + \lambda L)[y]$, in particular it minimizes $F[y]$ among the smaller class of those functions y for which $L[y]$ takes on some prescribed value; the convexity proof goes by checking the simple sufficient condition.

For the chain in a centrally symmetric potential $V(r)$, with length $2l$, one has

$$F[r] := \int_{\varphi_0}^{\varphi_1} V(r)\sqrt{r^2+\dot{r}^2}\,d\varphi = \int_0^{2l} V(r(t))\,dt,$$

where t is the arclength: $dt = \sqrt{r^2 + \dot{r}^2}\, d\varphi$. The constraint $L[r] = 2l$ turns into

$$\varphi_1 - \varphi_0 = \int_0^{2l} \frac{d\varphi(t)}{dt}\, dt = \int_0^{2l} \frac{1}{r(t)} \sqrt{1 - r'(t)^2}\, dt,$$

since $\frac{r'^2}{\dot{r}^2} = \varphi'^2 = \frac{1}{r^2+\dot{r}^2}$. Therefore, $\tilde{f}(t, r, p) = V(r) + \lambda\sqrt{1-p^2}/r$ in this case. Fixing r, it is clear, that $\tilde{f}(r, \cdot)$ cannot be convex, unless $\lambda \leq 0$. On the other hand, fixing $p = 0$, and assuming V to be concave, for $r \mapsto \tilde{f}(r, 0) = V(r) + \lambda/r$ to be convex, one would necessarily need $\lambda > 0$.

This argument includes in particular the potentials $-1/r^n$ and $\log(r)$.

In the case of the classical catenary considered by Troutman, [14, p. 76ff], the corresponding functional is $\tilde{f}(y, y') = y + \lambda\sqrt{1 - y'^2}$, and λ is indeed negative for chains, so the method works.

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