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Functional Analysis

Excercise Sheet 12

Problem 1 (Characterisation of the norm & Boundedness of weakly convergent sequences). Let $(X, \|\cdot\|)$ be a normed space.

a) Prove for all $x \in X$ that
$$\|x\| = \sup_{\substack{f \in X' \\ \|f\|_{X'} \leq 1}} |f(x)|.$$

b) Let $x \in X$ and $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ be such that
$$\lim_{n\to\infty} f(x_n) = f(x) \quad \forall f \in X'.$$
Prove that $\{x_n\}_{n\in\mathbb{N}}$ is bounded.

[5+5 Points]

Problem 2 (Weakly convergent sequences).

a) Let $\mathcal{H}$ be a separable Hilbert space. Prove that every ONB $\{e_n\}_{n\in\mathbb{N}}$ of $\mathcal{H}$ converges weakly to 0.

b) Let $f \in L^2(\mathbb{R})$. Prove that the following sequences $\{f_n\}_{n\in\mathbb{N}}$ converge weakly to 0 in $L^2(\mathbb{R})$:

i) $f_n(x) := f(x-n)$ ("wandering out to infinity").

ii) $f_n(x) := \frac{1}{n} f(\frac{x}{n^2})$ ("pancake").

iii) $f_n(x) := e^{inx} f(x)$ ("oscillation to death").

[2+(2+2+4) Points]

Problem 3 (The unit ball as a countable intersection of half-planes). Let X be a real normed space. We call a set $\{x \in X \mid \phi(x) \leq 1\}$, with $\phi \in X'$, a *half-space*. Prove that the closed unit ball $\mathcal{B}_1 := \{x \in L^2([0,1], \mathbb{R}) \mid \|x\| \leq 1\}$ can be represented as the intersection of countably many half-spaces, i.e. there exists $\{\phi_n\}_{n\in\mathbb{N}} \subseteq L^2([0,1], \mathbb{R})'$ such that
$$\mathcal{B}_1 = \bigcap_{n\in\mathbb{N}} \{x \in L^2([0,1], \mathbb{R}) \mid \phi_n(x) \leq 1\}.$$

[10 Points]

Problem 4 (The LIM-functional). In the following let $\ell^\infty := \ell^\infty(\mathbb{N}, \mathbb{R})$ (i.e. the $\mathbb{R}$-vector space of real-valued bounded sequences).

a) Prove that there exists a function $\mathrm{LIM} : \ell^\infty \to \mathbb{R}$ such that

   i) $\mathrm{LIM}(z + \alpha w) = \mathrm{LIM}(z) + \alpha\, \mathrm{LIM}(w)$ for all $z, w \in \ell^\infty$, $\alpha \in \mathbb{R}$,

   ii) $\liminf_{n\to\infty} z_n \leq \mathrm{LIM}(z) \leq \limsup_{n\to\infty} z_n$ for all $z \in \ell^\infty$,

   iii) $\mathrm{LIM}(z) = \lim_{n\to\infty} z_n$ for all $z \in c \subseteq \ell^\infty$.

b) What are the possible values of $\mathrm{LIM}(x)$ for $x = (x_n)_{n\in\mathbb{N}}$, where $x_n := (-1)^n$?

c) Find the set $\{(\mathrm{LIM}(x), \mathrm{LIM}(y)) \in \mathbb{R}^2 \mid \mathrm{LIM} \text{ satisfies i)-iii) in a)}\}$ for x as in b) and $y = (0, 1, 0, 1, 0, \dots)$.

Remark: On Excercise sheet 8, Problem 4 we have seen that not all elements of $(\ell^\infty)'$ are represented by an element of ℓ^1. This excercise shows how to study some of those elements not being represented by an element of ℓ^1. [4+4+2 Points]