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Functional Analysis

Excercise Sheet 11

Remaining solution

Problem 4 (Inverse mapping and closed graph theorem)**.** (Updated) Let X, Y, Z be Banach spaces and let $J : Y \to Z$ be an injective bounded linear map with closed range. Let $T : X \to Y$ be a linear map such that JT is bounded.

a) Prove that T is bounded.

b) Is it necessary that J has closed range for T to be bounded?

[10+0 Points]

Proof.

a) First observe that $R(J)$ is complete, since it is closed by assumption and hence complete as a closed subset of the Banach space Z. Then $J : Y \to R(J)$ is bijective and $J \in \mathcal{B}(Y, R(J))$. By the inverse mapping theorem follows that $J^{-1} \in \mathcal{B}(R(J), Y)$ is bounded. Hence $T = J^{-1}(JT)$ is bounded as the composition of bounded operators.

b) No: Let $((x_n, Tx_n))_n \subseteq \Gamma(T)$ and $(x, y) \in X \times Y$ such that $(x_n, Tx_n) \to (x, y)$ as $n \to \infty$. We have that $(JT)(x_n) \to (JT)(x)$ as $n \to \infty$, since JT is bounded (continuous) and $x_n \to x$ as $n \to \infty$. On the other hand, we have that $J(Tx_n) \to J(y)$ as $n \to \infty$, since J is bounded (continuous) and $Tx_n \to y$ as $n \to \infty$. Thus $J(y) = J(Tx)$ and hence, since J is injective, it follows that $y = Tx$. Hence the graph $\Gamma(T)$ is closed and T bounded by the closed graph theorem.

□