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Functional Analysis

Excercise Sheet 10

Remaining solution

Problem 1 (Finite rank operators are compact)**.** Let X, Y be normed spaces. A linear operator $T : X \to Y$ is called *finite-rank* if $\operatorname{rank}(T) := \dim R(T) < \infty$. Prove that a bounded linear operator is compact if it is finite-rank. [5 Points]

Proof. Let $A \subseteq X$ be bounded. Then there exists $R > 0$ s.t. $\|x\| < R$ for all $x \in A$. Since T is bounded we have for $x \in A$ that $\|Tx\| \leq \|T\| \, \|x\| < \|T\| \, R$. Thus

$$TA = \{Tx \mid x \in A\} \subseteq B_{\|T\|R}(0) \subset \overline{B_{\|T\|R}(0)}$$

is bounded and hence $\overline{TA}$ as well, as $\overline{TA}$ is the smallest closed set containing TA. Clearly $\overline{TA}$ is closed (observe that A and TA are sets and not spaces in general).

Claim: A subset $S \subseteq V$ of a finite dimensional normed $\mathbb{K}$-vectorspace V is compact *iff* it is closed and bounded.
Proof: From linear algebra we know that V is isometrically isomorphic to $\mathbb{K}^d$, where $d = \dim V < \infty$. Let $\varphi : V \to \mathbb{K}^d$ be such an isometric isomorphism. Since φ is in particular an homeomorphism, $\varphi(S) \subseteq \mathbb{K}^d$ is $\{$ closed, bounded, compact $\}$ *iff* S is. In $\mathbb{K}^d$ we know by Heine-Borel, that $\varphi(S)$ is compact *iff* it is closed and bounded. □

With $S := \overline{TA} \subseteq V := R(T)$ follows by the claim that T is compact. □

Alternative proof: Recall that an operator $T : X \to Y$ is compact *iff* for every bounded sequence $\{x_n\}_n \subseteq X$ the sequence $\{Tx_n\}_n \subseteq Y$ has a convergent subsequence.

Let $N := \operatorname{rank}(T) < \infty$ and $\{e_j\}_{j=1}^N$ be a basis of $V := R(T)$. Consider the 1-norm on V w.r.t. $\{e_j\}_{j=1}^N$, i.e. $\|y\|_1 := \sum_{j=1}^N |\alpha_j|$ for all $y = \sum_{j=1}^N \alpha_j e_j$, where $\{\alpha_j\}_{j=1}^N = \{\alpha_j(y)\}_{j=1}^N \subseteq \mathbb{C}^N$. Since all norms in finite-dimensional vector spaces are equivalent, there exist $c, C > 0$ such that $c \|y\|_Y \leq \|y\|_1 \leq C \|y\|_Y$ for all $y \in V \subseteq Y$.
Let $(x_n)_n \subseteq X$ be a bounded sequence in X, wlog $\|x_n\|_X \leq 1$ for all $n \in \mathbb{N}$. Let $\{\alpha_j^{(n)}\}_{j=1}^N$ denote the coordinates of $y_n := Tx_n \in V$ w.r.t. $\{e_j\}_{j=1}^N$. Then we have for all $j = 1, \dots, N$ that

$$|\alpha_j^{(n)}| \leq \sum_{j=1}^N |\alpha_j^{(n)}| = \|Tx_n\|_1 \leq C \|Tx_n\|_Y \leq \|T\|_{B(X,Y)} \|x_n\|_X \leq \|T\|_{B(X,Y)} \, .$$

Hence, for each j, $(\alpha_j^{(n)})_n$ is a bounded sequence in $\mathbb{C}$ and therefore admits a convergent subsequence. Thus we can iteratively extract convergent subsequences

$$\alpha_1^{(\tilde{n}_k)} \to \alpha_1, \quad \alpha_2^{(\tilde{n}_{k_l})} \to \alpha_2, \quad \dots .$$

such that after N steps, each component converges to some $\alpha_j \in \mathbb{C}$. Denoting this subsequence by $(\alpha^{(n_k)})_k$ and $y := \sum_j \alpha_j e_j$, we find that

$$c\,\|Tx_{n_k} - y\|_Y \leq \|Tx_{n_k} - y\|_1 = \sum_{n=1}^{N} |\alpha_j^{(n_k)} - \alpha_j| \to 0.$$

Thus for every bounded sequence $\{x_n\}_n \subseteq X$ the sequence $\{Tx_n\}_n \subseteq Y$ has a convergent subsequence. Hence T is compact. $\square$