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Functional Analysis

Excercise Sheet 8

Remaining solution

Problem 4 (Dual of ℓ^1 and ℓ^∞)**.**

a) Prove that $(\ell^1)' \cong \ell^\infty$, i.e. ℓ^∞ is isometrically isomorphic to the dual of ℓ^1.

b) Let $\{e_n\}_{n\in\mathbb{N}}$ be given by $e_n := (0,\dots,0,\overset{n\text{-th position}}{\overline{1}},0,\dots)$ (i.e. $(e_n)_k := \delta_{nk}$ for $k \in \mathbb{N}$) and let $J : (c_0)' \to (\ell^\infty)'$ be the linear map defined by $(Jg)(x) := \sum_n g(e_n)x_n$ for all $x \in \ell^\infty$ and $g \in (c_0)'$.

Prove that $(f(e_n))_{n\in\mathbb{N}} \in \ell^1$ for all $f \in (c_0)'$, that J is a linear isometry, and that every $f \in (\ell^\infty)'$ has a unique representation $f = f_1 + f_2$, where $f_1 \in J((c_0)')$ and $f_{2|c_0} = 0$.

Remark: This proves that $(\ell^\infty)' \cong \ell^1 \oplus (c_0)^\perp$, where $\oplus$ denotes the direct sum and $(c_0)^\perp := \{\eta \in (\ell^\infty)' \mid \eta(x) = 0\ \forall x \in c_0\}$. From the Hahn-Banach theorem that will be proved later will follow that $(c_0)^\perp \neq \emptyset$ and hence that ℓ^1 can be embedded isometrically into but *not* onto $(\ell^\infty)'$.

[4+6 Points]

Proof. b) • Let $f \in (c_0)'$ and for $n \in \mathbb{N}$ define $s_n := \overline{f(e_n)}/|f(e_n)|$ if $f(e_n) \neq 0$ and $s_n := 0$ otherwise. Then we have for all $N \in \mathbb{N}$ by the linearity of f that

$$\sum_{n=1}^{N} |f(e_n)| = \sum_{n=1}^{N} s_n f(e_n) = f\left(\sum_{n=1}^{N} s_n e_n\right)$$
$$\leq \|f\|_{(c_0)'} \left\|\sum_{n=1}^{N} s_n e_n\right\|_\infty = \|f\|_{(c_0)'} \max_{1\leq n\leq N} |s_n| \leq \|f\|_{(c_0)'},$$

and hence $\|(f(e_n))_{n\in\mathbb{N}}\|_1 = \sum_{n=1}^\infty |f(e_n)| \leq \|f\|_{(c_0)'} < \infty$, i.e. $(f(e_n))_{n\in\mathbb{N}} \in \ell^1$.

• J is clearly linear. We have to prove that $\|Jf\|_{(\ell^\infty)'} = \|f\|_{(c_0)'}$ for all $f \in (c_0)'$. As above follows that $\|(f(e_n))_n\|_1 \leq \|f\|_{(c_0)'}$ for $f \in (c_0)'$ and hence by Hölder's inequality that

$$|(Jf)(x)| = \left|\sum_{n=1}^{\infty} f(e_n)x_n\right| \leq \|(f(e_n))_n\|_1 \|x\|_\infty \leq \|f\|_{(c_0)'} \|x\|_\infty$$

for $f \in (c_0)'$ and $x \in \ell^\infty$, i.e. $\|Jf\|_{(\ell^\infty)'} \leq \|f\|_{(c_0)'}$. For the reverse inequality observe first, that, for $f \in (c_0)'$ and $x \in c_0$, we have that

$$(Jf)(x) = \sum_{n=1}^{\infty} f(e_n)x_n = \lim_{N\to\infty} \sum_{n=1}^{N} f(e_n)x_n = \lim_{N\to\infty} f\left(\sum_{n=1}^{N} x_n e_n\right) = f(x),$$

since $\sum_{n=1}^{N} x_n e_n \to x$ wrt. $\|\cdot\|_\infty$ (which in general is not true for $x \in \ell^\infty$). Then

$$\|Jf\|_{(\ell^\infty)'} = \sup_{0\neq x\in\ell^\infty} \frac{|(Jf)(x)|}{\|x\|_\infty} \geq \sup_{0\neq x\in c_0} \frac{|(Jf)(x)|}{\|x\|_\infty} = \sup_{0\neq x\in c_0} \frac{|f(x)|}{\|x\|_\infty} = \|f\|_{(c_0)'}.$$

Hence J is an isometry.

• Let $f \in (\ell^\infty)'$. Then $f_{|c_0} \in (c_0)'$, since $\left\|f_{|c_0}\right\|_{(c_0)'} = \sup_{0\neq x\in c_0} \frac{|f_{|c_0}(x)|}{\|x\|_\infty} \leq \|f\|_{(\ell^\infty)'}$. Thus $f_1 := J(f_{|c_0}) \in J((c_0)')$ and $f_2 := f - f_1 \in (\ell^\infty)'$ and we have for $x \in c_0$ that $f_2(x) = \sum_{n=1}^{\infty} f_{|c_0}(e_n)x_n - f(x) = f\left(\sum_{n=1}^{\infty} x_n e_n\right) - f(x) = 0$, i.e. $f_{2|c_0} = 0$.

• It remains to show uniqueness: Let $f_1, g_1 \in J((c_0)')$ and $f_2, g_2 \in (\ell^\infty)'$ s.t. $f_{2|c_0} = g_{2|c_0} = 0$ and $f = f_1 + f_2 = g_1 + g_2$. This directly implies that $f_{1|c_0} = g_{1|c_0}$.

Let $h \in J((c_0)')$ and $\tilde{h} \in (c_0)'$ s.t. $h = J\tilde{h}$. Then for $x \in \ell^\infty$:

$$\begin{aligned}(Jh_{|c_0})(x) &= \sum_{n=1}^{\infty} h_{|c_0}(e_n)x_n = \sum_{n=1}^{\infty} h(e_n)x_n = \sum_{n=1}^{\infty} (J\tilde{h})(e_n)x_n \\ &= \sum_{n=1}^{\infty}\sum_{l=1}^{\infty} \tilde{h}(e_l)(e_n)_l x_n = \sum_{n=1}^{\infty} \tilde{h}(e_n)x_n = (J\tilde{h})(x).\end{aligned}$$

This shows that $Jh_{|c_0} = J\tilde{h}$ and therefore $\tilde{h} = h_{|c_0}$ as J is injective.

Thus $f_1 = J(f_{1|c_0}) = J(g_{1|c_0}) = g_1$ and hence $f_2 = f - f_1 = f - g_1 = g_2$.

□