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Functional Analysis
Excercise Sheet 8

Problem 1 (Norm of functionals)**.** Compute the norm of the following linear functionals, where the spaces are equipped with their usual norm:

a) $\phi : C([-1,1]) \to \mathbb{K}, f \mapsto \int_{-1}^{1} x f(x) \mathrm{d}x$.

b) $\phi : \ell^2 \to \mathbb{K}, x = (x_n)_n \mapsto \sum_{n=1}^{\infty} \frac{x_n}{n}$.

c) $\phi : \ell^\infty \to \mathbb{R}$, s.t. $\phi(x) \geq 0$ for all $x \in \{(x_n)_n \in \ell^\infty \mid x_n \geq 0 \text{ for all } n \in \mathbb{N}\}$, and s.t. $\phi(\mathbf{1}) = 2016.0613$, where $\mathbf{1} = (1, 1, \dots) \in \ell^\infty$.

Here, $\ell^\infty \coloneqq \ell^\infty(\mathbb{N}, \mathbb{R}) \coloneqq \{(x_n)_{n\in\mathbb{N}} \mid x_n \in \mathbb{R}\ \forall n \in \mathbb{N} \wedge \sup_{n\in\mathbb{N}} |x_n| < \infty\}$.

[3+3+4 Points]

Problem 2 (Relations between functionals and their kernel)**.** Let $(X, \|\cdot\|)$ be a normed space and let ϕ be a bounded linear functional on X.

a) Prove that $|\phi(x)| = \|\phi\|_{X'} \cdot \operatorname{dist}(x, \ker\phi)$ for all $x \in X$.

b) Prove that $|\phi(x) - a| = \|\phi\|_{X'} \cdot \operatorname{dist}(x, \phi^{-1}(\{a\}))$ for all $x \in X$, $a \in \mathbb{K}$ if $\phi \neq 0$.

Hint: Prove that $\phi^{-1}(\{a\}) = \frac{aw}{\phi(w)} + \ker\phi$ for some $w \in X$.

c) Prove that the following statements are equivalent if $\phi \neq 0$:

(i) ϕ does not attain its norm (i.e. $\forall x \in X \setminus \{0\} : |\phi(x)| < \|\phi\|_{X'} \|x\|$).

(ii) There is no $x \in X$ s.t. $\|x\| = 1$ and $\operatorname{dist}(x, \ker\phi) = 1$.

(iii) The distance from $x \in X \setminus (\ker\phi)$ to $\ker\phi$ is never attained (i.e. $\forall x_0 \in \ker\phi$: $\operatorname{dist}(x, \ker\phi) < \|x - x_0\|$).

d) Explain why (ii) does not contradict Riesz' Lemma.

[4+4+3+1 Points]

Problem 3 (Norm-distance between hyperplanes)**.**

a) Let $(X, \|\cdot\|)$ be a normed space, $\phi \in X'$ a non-zero bounded linear functional on X, and $\Pi_j \coloneqq \{x \in X \mid \phi(x) = a_j\}$ a hyperplane, where $a_j \in \mathbb{K}$, $j = 1, 2$.

Prove that $\operatorname{dist}(\Pi_1, \Pi_2) = \frac{|a_1 - a_2|}{\|\phi\|_{X'}}$.

b) Compute the distance in $(\mathbb{R}^3, \|\cdot\|_2)$ between the parallel hyperplanes

$$\Pi_1 \coloneqq \{x \in \mathbb{R}^3 \mid a \cdot x = d_1\} \quad \text{and} \quad \Pi_2 \coloneqq \{x \in \mathbb{R}^3 \mid a \cdot x = d_2\}, \tag{1}$$

where $a \in \mathbb{R}^3 \setminus \{0\}$ and $d_1, d_2 \in \mathbb{R}$.

c) Compute the distance in $(C([0,1]), \|\cdot\|_\infty)$ between the two subsets

$$E_1 := \left\{ f \in C([0,1]) \,\middle|\, \int_0^1 f(x)\mathrm{d}x = 1 \right\} \text{ and } E_2 := \left\{ f \in C([0,1]) \,\middle|\, \int_0^1 f(x)\mathrm{d}x = -1 \right\}. \tag{2}$$

[4+2+2 Points]

Problem 4 (Dual of ℓ^1 and ℓ^∞).

a) Prove that $(\ell^1)' \cong \ell^\infty$, i.e. ℓ^∞ is isometrically isomorphic to the dual of ℓ^1.

b) Let $\{e_n\}_{n\in\mathbb{N}}$ be given by $e_n := (0, \dots, 0, \overbrace{1}^{n\text{-th position}}, 0, \dots)$ (i.e. $(e_n)_k := \delta_{nk}$ for $k \in \mathbb{N}$) and let $J : (c_0)' \to (\ell^\infty)'$ be the linear map defined by $(Jg)(x) := \sum_n g(e_n)x_n$ for all $x \in \ell^\infty$ and $g \in (c_0)'$.

Prove that $(f(e_n))_{n\in\mathbb{N}} \in \ell^1$ for all $f \in (c_0)'$, that J is a linear isometry, and that every $f \in (\ell^\infty)'$ has a unique representation $f = f_1 + f_2$, where $f_1 \in J((c_0)')$ and $f_{2|c_0} = 0$.

Remark: This proves that $(\ell^\infty)' \cong \ell^1 \oplus (c_0)^\perp$, where $\oplus$ denotes the direct sum and $(c_0)^\perp := \{\eta \in (\ell^\infty)' \mid \eta(x) = 0 \ \forall x \in c_0\}$. From the Hahn-Banach theorem that will be proved later will follow that $(c_0)^\perp \neq \emptyset$ and hence that ℓ^1 can be embedded isometrically into but *not* onto $(\ell^\infty)'$.

[4+6 Points]

***Deadline: June 13, 2016 14:00**, for details see http://www.math.lmu.de/˜gottwald/16FA/.*