LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

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Functional Analysis

Excercise Sheet 5

Remaining solutions

Problem 1 (Contractions, Compactness & Completeness).

a) Let (X, d) be a non-empty compact metric space and let $\Phi : X \to X$ be such that

$$d(\Phi(x), \Phi(y)) < d(x, y), \text{ for all } x, y \in X, x \neq y. \tag{1}$$

Prove that Φ has a unique fixed point $x_0 = \lim_{n\to\infty} \Phi^n(x)$, where $x \in X$ arbitrary.

b) Find a non-empty compact metric space (X, d) and a function $\Phi : X \to X$ such that $d(\Phi(x), \Phi(y)) \leq d(x, y)$ for all $x, y \in X$, $x \neq y$ and Φ does *not* have a fixed point. What is the difference to a)?

c) Prove that there does not exist a surjective contraction from a compact metric space with more than one elements onto itself.

d) Let (X, d) be a non-empty complete metric space and let $\Phi : X \to X$ be such that Φ^m is a contraction for some $m \in \mathbb{N}$. Prove that Φ has a unique fixed point.

e) Let (X, d) be a non-empty metric space such that any contraction $\Phi : E \to E$ on any non-empty closed subset $E \subseteq X$ has a fixed point. Prove that (X, d) is complete.

[3+2+2+2+3 Points]

Proof. a) *Uniqueness*: Assume there exists $x, y \in X$ with $x \neq y$ s.t. $\Phi(x) = x$ and $\Phi(y) = y$. Then $d(x, y) = d(\Phi(x), \Phi(y)) < d(x, y)$. Contradiction! Thus the fixed point is unique (if it exists at all).

Existence: Consider the the function $F : X \to \mathbb{R}_+, x \mapsto d(x, \Phi(x))$. Using the triangle inequality and (1) we get for $x \neq y$, that

$$F(x) = d(x, \phi(x)) \leq d(x, y) + d(y, \phi(y)) + d(\phi(y), \phi(x)) < 2\, d(x, y) + F(y),$$

and thus by symmetry, that $|F(x) - F(y)| \leq 2\, d(x, y)$ for all $x, y \in X$. This implies that F is continuous. Since X is compact, there exists $x_0 \in X$ such that $F(x_0) \leq F(x)$ for all $x \in X$. Assume $x_0 \neq \Phi(x_0)$. Then by (1) $F(\Phi(x_0)) = d(\Phi(x_0), \Phi(\Phi(x_0))) < d(x_0, \Phi(x_0)) = F(x_0)$. This contradicts, that x_0 minimizes F. Thus x_0 is a fixed point.

Hint: Consider the non-empty compact metric space $([0, 1], d_{Eucl})$ and the map $\Phi : [0, 1] \to [0, 1], x \mapsto \frac{1}{2}x^2$. It satisfies $|\Phi(x) - \Phi(y)| = \frac{1}{2}|x^2 - y^2| = \frac{1}{2}|x + y||x - y| <$

$|x-y|$ for all $x, y \in [0,1]$ with $x \neq y$. But Φ is *not* a contraction, since its Lipschitz constant $L := \sup_{0 \le x < y \le 1} \frac{|\Phi(x)-\Phi(y)|}{|x-y|} = \sup_{0 \le x < y \le 1} |x+y| = 1$. This shows in particularly, that (1) is not strong enough to get a Lipschitz constant $L < 1$. Thus the Banach fixed-point theorem can not be used here.

It remains to show, that $x_0 = \lim_{n\to\infty} \Phi^n(x)$, where $x \in X$ is arbitrary. Let $x \in X$ and $x_n := \Phi^n(x)$ for all $n \in N$. Wlog $x_n \neq x_0$ for all $n \in N$ (otherwise we are done).

Since X is compact and, since X is a metric space, sequentially compact, the sequence $\{x_n\}_n$ has a convergent subsequence, say $x_{n_l} \to y \in X$ for $l \to \infty$. Since Φ is (Lipschitz-)continuous it is also sequentially continuous and thus $\Phi(x_{n_l}) \to \Phi(y)$ for $l \to \infty$.

By (1) we have $0 < d(x_{n+1}, x_0) = d(\Phi(x_n), \Phi(x_0)) < d(x_n, x_0)$ for all $n \in \mathbb{N}$. Thus $\{d(x_n, x_0)\}_n$ is a strictly decreasing sequence bounded from below by 0. Thus it is convergent so some value $a := \lim_{n\to\infty} d(x_n, x_0) \ge 0$. Thus the subsequences $\{d(x_{n_l}, x_0)\}_n$ and $\{d(x_{n_l+1}, x_0)\}_n$ also converge to a, i.e. $\lim_{l\to\infty} d(x_{n_l}, x_0) = a$ and $\lim_{l\to\infty} d(x_{n_l+1}, x_0) = a$.

By the continuity of d (i.e. $x \mapsto d(x, x_0)$) we have that $d(x_{n_l}, x_0) \to d(y, x_0) = a$ and $d(x_{n_l+1}, x_0) = d(\Phi(x_{n_l}), x_0) \to d(\Phi(y), x_0) = a$ for $l \to \infty$.

If $y \neq x_0$ we get in total by (1):

$$0 < d(y, x_0) = a = d(\Phi(y), x_0) = d(\Phi(y), \Phi(x_0)) < d(y, x_0).$$

Contradiction! Thus $y = x_0$ and $a = d(y, x_0) = 0$. Hence $d(x_n, x_0) \to 0$ i.e. $\Phi^n(x) \to x_0$ for $n \to \infty$.

b) Equipp $X := \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1\}$ with the Euclidian metric d and let Φ be the rotation by $\pi/2$, i.e. $\Phi : X \to X, (x,y) \mapsto (-y, x)$. Then X is bounded and closed and hence compact and for all $(x,y), (x',y') \in X$ we have $d(\Phi(x,y), \Phi(x',y')) = d((-y,x), (-y',x')) = d((x,y),(x',y'))$. Assume that $\Phi(x,y) = (-y,x) \overset{!}{=} (x,y)$ for some $(x,y) \in X$. Then $x = y = 0$. Contradiction, since $(0,0) \notin X$. Thus Φ does not have a fixed point.

In contrast to a) equality was allowed (and used) in $d(\Phi(x), \Phi(y)) < d(x,y)$.

c) If $|X| = 1$ and d any metric on X, then (X,d) is a non-empty compact metric space and the identity map $Id : X \to X$ a surjective contraction, because $d(Id(x), Id(x)) = 0 \le \frac{1}{2} d(x,x) = 0$.

Let (X,d) with $|X| > 1$ be a compact metric space and assume that Φ is a surjective contraction. By Tutorial 4, Problem 2(ii) X is bounded and thus $\operatorname{diam}(X) := \sup_{x,y\in X} d(x,y) < \infty$. Since $|X| > 1$ we also have $\operatorname{diam}(X) > 0$. As Φ is surjective we have $\Phi(X) = X$ and thus $\operatorname{diam}(\Phi(X)) = \operatorname{diam}(X)$.

As Φ is a contraction with constant $L < 1$ we have on the other hand that

$$\begin{aligned}\operatorname{diam}(\Phi(X)) &= \sup_{p,q\in\Phi(X)} d(p,q) = \sup_{x,y\in X} d(\Phi(x), \Phi(y)) \\ &\le L \sup_{x,y\in X} d(x,y) = L \operatorname{diam}(X) < \operatorname{diam}(X) = \operatorname{diam}(\Phi(X)).\end{aligned}$$

Contradiction!

d) Let x_0 be the unique fixed point for the contraction Φ^m, which exists by the Banach fixed-point theorem, since X is complete. Due to $\Phi^m(\Phi(x_0)) = \Phi(\Phi^m(x_0)) = \Phi(x_0)$ $\Phi(x_0)$ is also a fixed point for Φ^m. By uniqueness follows $x_0 = \Phi(x_0)$. Thus Φ has a fixed point. Assume that y_0 is another fixed point for Φ. Then $y_0 = \Phi(y_0) = \Phi^2(y_0) = \ldots = \Phi^m(y_0)$. Thus y_0 is a fixed point of Φ^m and by uniqueness follows $y_0 = x_0$.

e) Let (X, d) be a non-empty metric space such that any contraction $\Phi : E \to E$ on any non-empty closed subset $E \subseteq X$ has a fixed point. Prove that (X, d) is complete.

f) Let $\{x_n\}_n$ be a Cauchy sequence in (X, d). Wlog it is not eventually constant, since we are done otherwise. Wlog $\mathbb{N} \ni n \mapsto x_n$ is injective (i.e. no value of the sequence appears more than once). Otherwise we could choose a subsequence, since the sequence is not eventually constant. Assume that $\{x_n\}_n$ is *not* convergent.

Consider $I : X \to \mathbb{R}, x \mapsto \inf\{d(x, x_n) \mid n \in \mathbb{N} \text{ s.t. } x_n \neq x\}$. Then $I(x) > 0$ for all $x \in X$, since $\{x_n\}_n$ is not convergent [$I(x) = 0 \iff x_{n_k} \to x$ for $k \to \infty$ for some subsequence $\{x_{n_k}\}_k$ (cf. Lemma 1.47)].

Inductively we define a subsequence of $\{x_n\}_n$: Let $N_1 := 1$ and $y_1 := x_{N_1}$. Next assume that N_j and y_j are defined for some $j \in \mathbb{N}$. Since $\{x_n\}_n$ is Cauchy, there exists $N_{j+1} > N_j$ s.t. $d(x_k, x_l) \leq \frac{1}{2} I(y_j)(= \varepsilon_j)$ for all $k, l \geq N_{j+1}$. Let $y_{j+1} := x_{N_{j+1}}$.

Let $E := \{y_j \mid j \in \mathbb{N}\}$ and $\Phi : E \to E, y_j \mapsto \Phi(y_j) := y_{j+1}$. Then Φ is a contraction, since for $k > l$ we have $N_{k+1} \geq N_{l+1}$ and thus by definition of $\{y_j\}_j$ and of I:

$$d(\Phi(y_k), \Phi(y_l)) = d(x_{N_{k+1}}, x_{N_{l+1}}) \leq \frac{1}{2} I(y_l) \leq \frac{1}{2} d(y_l, y_k).$$

If E was closed, then Φ would by assumption have a fixed point $y = y_k \in E$ for some $k \in \mathbb{N}$, i.e. $y_k = y = \Phi(y) = \Phi(y_k) = y_{k+1} \neq y_k$, in contradiction to the assumption that $n \mapsto x_n$ is injective. Thus E is not closed and $\overline{E} \supsetneq E$.

Let $y \in \overline{E} \setminus E \neq \emptyset$. Then y is a limit point (Lemma 1.10) and there exists a sequence $\{y_{j_k}\}_k$ in E converging to y. Thus we have a convergent subsequence $\{y_{j_k}\}_k$ of the Cauchy sequence $\{x_n\}_n$. By Lemma 1.47 $\{x_n\}_n$ then is convergent in contradiction to the assumption at the beginning. Thus $\{x_n\}_n$ is convergent and hence (X, d) complete.

$\square$

Problem 3 (Product space & Compactness)**.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be compact topological spaces. Prove that $(X \times Y, \mathcal{T}_{X \times Y})$ is compact, where $\mathcal{T}_{X \times Y}$ is the product topology on $X \times Y$. [10 Points]

Proof. First we need the following

Claim: *Let $(Z, \mathcal{T})$ be a topological space and $\mathcal{B}$ a base for $\mathcal{T}$. Then Z is compact, if every open cover of Z by sets from $\mathcal{B}$ has a finite subcover.* (Cf. Tutorial 4, Problem 2(iii))

Proof of claim: Let $Z = \bigcup_{i \in I} U_i$ be an open cover of Z with $U_i \in \mathcal{T}$ for all $i \in I$. For each $z \in Z$ choose $B_z \in \mathcal{B}$ and $i_z \in I$ s.t. $z \in B_z \subseteq U_{i_z}$. This is possible, because $\mathcal{B}$ is a base. Then $Z = \bigcup_{z \in Z} B_z$ is an open cover of Z by sets of $\mathcal{B}$, which by assumption admits a finite subcover $Z = \bigcup_{j=1}^{N} B_{z_j}$. Since $B_{z_j} \subseteq U_{i_{z_j}}$ for $j = 1..N$ we have a finite subcover $Z = \bigcup_{j=1}^{N} U_{i_{z_j}}$. Thus Z is compact. $\square$

Now we can prove, that $X \times Y$ is compact in the product topology.
Let $\mathcal{M} := \{R_i \times S_i \mid i \in I\}$ be an open cover of $X \times Y$ by rectangles, i.e. $R_i \in \mathcal{T}_X$ and $S_i \in \mathcal{T}_Y$ for all $i \in I$. By the above claim it is enough to show that there exists a finite subcover, since rectangles form a basis of the product topology.
Let $x \in X$. For all $y \in Y$ exists $R_y \times S_y \in \mathcal{M}$ s.t. $(x, y) \in R_y \times S_y$. Note, that for all $y \in Y$ R_y is a nbhd. of x and S_y a nbhd. of y. Then $\{S_y \mid y \in Y\}$ is an open cover of Y. Since Y is compact, there exists $y_1, \ldots, y_N$ s.t. $\bigcup_{j=1}^{N} S_{y_j} \supseteq Y$. In addition $U_x := \bigcap_{j=1}^{N} R_{y_j}$ is a nbhd of x in X and $U_x \times Y \subseteq \bigcup_{j=1}^{N}(R_{y_j} \times S_{y_j})$.
Thus $\{U_x \mid x \in X\}$ is an open cover of X, which due to compactness has a finite subcover $\{U_{x_1}, \ldots, U_{x_M}\}$. Finally we have that

$$X \times Y \subseteq \bigcup_{i=1}^{M} U_{x_i} \times Y \subseteq \bigcup_{i=1}^{M} \bigcup_{j=1}^{N_{x_i}} (R_{y_j}^{x_i} \times S_{y_j}^{x_i}),$$

i.e. $\{(R_{y_j}^{x_i} \times S_{y_j}^{x_i} \mid i = 1, \ldots, M, \ \ i = 1, \ldots N_{x_i}\}$ is a finite subcover of $\mathcal{M}$ and hence $X \times Y$ compact. $\square$