LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

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FUNCTIONAL ANALYSIS

EXCERCISE SHEET 4

SOLUTION

Problem 1 (COMPACTNESS)**.**

a) Consider $A := (0, 1)$ as a subset of the metric space $(\mathbb{R}, d_{Eucl})$. Find an open cover of A which does not admit a finite subcover.

b) Consider $B := [0, 1] \cap \mathbb{Q}$ as a subset of the metric space $(\mathbb{Q}, d_{Eucl})$. Prove that B is closed and bounded in $\mathbb{Q}$ and find an open cover of B which does not admit a finite subcover.

c) Find a Hausdorff non-compact space, a finite compact non-Hausdorff space, and an infinite compact non-Hausdorff space.

[2+4+4 Points]

Proof. a) Let $U_n := (\frac{1}{n}, 1 - \frac{1}{n})$ for all $n > 2$. Let $x \in A$. Then there exists $n \in \mathbb{N}$ s.t. $0 < \frac{1}{n} < x < 1 - \frac{1}{n}$. Thus $x \in U_n$. Hence $\bigcup_{n=3}^{\infty} U_n \supseteq A$ and thus $\{U_n\}_{n \geq 3}$ is an open cover of A. Assume there exists a finite subcover $U_{n_1}, \dots, U_{n_N}$. Then $M := \max\{n_1, \dots n_N\} < \infty$ and $A \not\subseteq \bigcup_{j=1}^{N} U_{n_j} = (\frac{1}{M}, 1 - \frac{1}{M})$. Contradiction!

b) The intervall $[0, 1]$ is closed in $(\mathbb{R}, \mathcal{T}_{Eucl})$. By Sheet 1, Problem 3a) B is closed in $(\mathbb{R} \cap \mathbb{Q} = \mathbb{Q}, \mathcal{T}_{\mathbb{Q}})$, where $\mathcal{T}_{\mathbb{Q}}$ is the relative topology of $\mathbb{Q}$ wrt. $\mathcal{T}_{Eucl}$. By Tutorial 5 $\mathcal{T}_{\mathbb{Q}}$ is equal to the topology induced by $d_{Eucl|\mathbb{Q}\times\mathbb{Q}}$. Thus B is closed in $(\mathbb{Q}, d_{Eucl})$.

B is bounded, since $B \subseteq B_2(0) = \{y \in \mathbb{Q} \mid d(0, y) < 2 < \infty\}$.

Let $U_n := [(-1, \frac{1}{\sqrt{2}} - \frac{1}{n}) \cup (\frac{1}{\sqrt{2}} + \frac{1}{n}, 2)] \cap \mathbb{Q}$ for all $n > 2$. These sets are open in $(\mathbb{Q}, d_{Eucl})$ by analog arguments as above, since $(-1, \frac{1}{\sqrt{2}} - \frac{1}{n}) \cup (\frac{1}{\sqrt{2}} + \frac{1}{n}, 2)$ is open in $(\mathbb{R}, \mathcal{T}_{Eucl})$. Hence $\bigcup_{n=3}^{\infty} U_n = [(-1, 2) \setminus \{\frac{1}{\sqrt{2}}\}] \cap \mathbb{Q} = (-1, 2) \cap \mathbb{Q} \supseteq B$ and thus $\{U_n\}_{n\geq3}$ is an open cover of B, since $\frac{1}{\sqrt{2}} \notin \mathbb{Q}$. Assume there exists a finite subcover $U_{n_1}, \dots, U_{n_N}$. Then $M := \max\{n_1, \dots n_N\} < \infty$ and $B \not\subseteq \bigcup_{j=1}^{N} U_{n_j} = [(-1, \frac{1}{\sqrt{2}} - \frac{1}{M}) \cup (\frac{1}{\sqrt{2}} + \frac{1}{M}, 2)] \cap \mathbb{Q}$, since $[\frac{1}{\sqrt{2}} - \frac{1}{M}, \frac{1}{\sqrt{2}} + \frac{1}{M}] \cap \mathbb{Q} \neq \emptyset$ ($\mathbb{Q}$ is dense in $\mathbb{R}$). Contradiction!

c) • *Hausdorff, non-compact*: $(\mathbb{R}, d_{Eucl})$ is as a metric space Hausdorff and first countable. If it were compact, it were sequentially compact (cf. lecture) and every sequence had a convergent subsequence. But $\{n\}_n$ does not have a convergent subsequence. Thus $(\mathbb{R}, d_{Eucl})$ is non-compact.

- *finite, compact, non-Hausdorff*: Let $(X, \mathcal{T})$ be a finite topological space. Let $\{U_n\}_n$ be an open cover of X. Since $X = \{x_1, \dots x_N\}$ is finite there exist $n_1, \dots, n_N$ s.t. $x_i \in U_{n_i}$ for $i = 1, \dots, N$ and thus $\bigcup_{i=1}^N U_{n_i} \supseteq X$. Thus every finite space is compact, independent of the topology.

  Equipping X with the indiscrete or the co-finite topology makes it non-Hausdorff.
- *infinite, compact, non-Hausdorff*: $(X, \mathcal{T}_{indisc})$ with X infinite is compact and Hausdorff, since every open covering is already finite.

$\square$

Problem 2 (Homeomorphisms)**.**

a) Let X, Y be topological spaces and $f : X \to Y$ a continuous function.

   Prove that $f(X)$ is compact if X is compact.

b) Let $X := [0, 1)$ and $Y := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ be equipped with the relative topologies induced by the Euclidian topologies on $\mathbb{R}$ resp. $\mathbb{R}^2$.

   Prove that
$$\varphi : X \to Y, \theta \mapsto (\cos(2\pi\theta), \sin(2\pi\theta))$$
   is a continuous bijection, but not a homeomorphism. Are X and Y homeomorphic?

c) Let X be a compact space and let Y be a Hausdorff space.

   Prove or disprove: If $f : X \to Y$ is a continuous bijection then it is a homeomorphism.

d) Let X be a compact Hausdorff space and let Y be a compact space.

   Prove or disprove: If $f : X \to Y$ is continuous bijection then it is a homeomorphism.

[2+5+2+1 Points]

Proof.

a) Let $\bigcup_{i \in I} U_i$ be an open cover of $f(X) \subseteq Y$. Then $f^{-1}(U_i) \subseteq X$ is open for all $i \in I$, since f is continuous. Furthermore $\bigcup_{i \in I} f^{-1}(U_i)$ is an open cover of X. [Let $x \in X$, then $f(x) \in \bigcup_{i \in I} U_i$, i.e. $f(x) \in U_{i_0}$ for some $i_0 \in I$. Finally $x \in f^{-1}(U_{i_0})$ and thus $X \subseteq \bigcup_{i \in I} f^{-1}(U_i)$.] Since X is compact there exists $i_1, \dots i_N \in I$ s.t. $X \subseteq \bigcup_{j=1}^N f^{-1}(U_{i_j})$ and finally $f(X) \subseteq \bigcup_{j=1}^N U_i$. Thus $f(X)$ is compact.

   Hint: This implies that *if $\Phi : X \to Y$ is a homeomorphism, then X is compact* iff *Y is compact.* Indeed, if X is compact, then we have by a) and since Φ is surjective, that $\Phi(X) = Y$ is compact. On the other hand, if Y is compact, then by a) since Φ^{-1} is continuous and surjective, that $\Phi^{-1}(Y) = X$ is compact.

b)
   - φ *injective*: Let $\theta_1, \theta_2 \in X$ s.t. $\varphi(\theta_1) = \varphi(\theta_2)$. Then $\cos(2\pi\theta_1) = \cos(2\pi\theta_2)$ and $\sin(2\pi\theta_1) = \sin(2\pi\theta_2)$ and hence $\theta_1 = \theta_2$, since cos and sin are bijective on $[0, 2\pi)$ (cf. Analysis 1).
   - φ *surjective*: Let $(x, y) \in Y$. Then we find an angle $2\pi\theta$ with $\theta \in X$ between the positive x-axis and the radial line from 0 to (x, y) s.t. $\varphi(\theta) = (x, y)$.
   - φ *continuous*: That φ is continuous as a function from X to $\mathbb{R}^2$ follows from Analysis 2. Therefore it is also continuous as $\varphi : X \to Y$ wrt. to the relative topology of Y.

- φ^{-1} *not continuous*: We show, that φ^{-1} is not continuous at $x = (1, 0)$. Set $\varepsilon := \frac{1}{4}$ and let $\delta > 0$ be given. Wlog $\delta < 1$. Set $y := (1 - \frac{\delta^2}{4}, -\sqrt{\frac{\delta^2}{2} - \frac{\delta^4}{16}})$. Then $y^2 = 1$ and thus $y \in Y$. In addition we have $|x - y| = \frac{\delta}{\sqrt{2}} < \delta$. But on the other hand $|\varphi^{-1}(x) - \varphi^{-1}(y)| = |\varphi^{-1}(y)| > \frac{3}{4} > \varepsilon$. Thus φ^{-1} can not be continuous at x.
- Assume there exists a homeomorphism $\Phi : X \to Y$. By the hint in a) X is compact iff Y is compact. But X is not closed and hence not compact in $\mathbb{R}$, whereas Y is closed and bounded and hence compact in $\mathbb{R}^2$. Contradiction!

c) We only have to show, that $g := f^{-1} : Y \to X$ is continuous. Let $U \subseteq X$ be open. Then $A := X \setminus U$ is closed. $g^{-1}(U)$ is open iff $Y \setminus g^{-1}(U) = g^{-1}(X \setminus U) = g^{-1}(A) = f(A)$ is closed. By a) and as f is surjective, we have that $Y = f(X)$ is compact. By Prop. 1.44 follows, that $A \subseteq Y$ is compact as well. Since f is continuous we have by a proof similar to a) that $f(A)$ is compact. Since Y is Hausdorff we have due to Prop. 1.43 that $f(A)$ is closed.

d) Equip $X = [0, 1]$ with the Euclidian topology and $Y = [0, 1]$ with the indiscrete topology. Then X is compact and Hausdorff and Y is compact but not Hausdorff. Let f be the identity from X to Y. Then f is a bijection and continuous, since $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(Y) = X$ are both open in X. But f^{-1} is not continuous, since e.g. $f^{-1^{-1}}((1/3, 1/2))$ is not open in Y, although $(1/3, 1/2)$ is open in X.

□

Problem 3 ($\mathbb{Q}$ CAN BE OPEN).

a) Prove that $\mathbb{Q}$ is neither open nor closed in $(\mathbb{R}, d_{Eucl})$.

b) Prove that $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_0^+$ with

$$d(x, y) := |x - y| + \sum_{n=1}^{\infty} \frac{1}{2^n} \min \left\{ 1, \left| \frac{1}{\min_{j \leq n} |x - q_j|} - \frac{1}{\min_{j \leq n} |y - q_j|} \right| \right\},$$

where $\{q_j\}_{j \in \mathbb{N}}$ is an enumeration of $\mathbb{Q}$, is a metric on $\mathbb{R}$. (*Hint*: By convention $1/0 = \infty$, $|\infty - \infty| = 0$ and $|\infty - a| = |a - \infty| = \infty$ for all $a \in \mathbb{R}$.)

c) Prove that $\{q\}$ is open in $(\mathbb{R}, d)$ for all $q \in \mathbb{Q}$.

d) Prove that $\mathbb{Q}$ is open in $(\mathbb{R}, d)$.

[2+3+4+1 Points]

Proof. a) Assume that $\mathbb{Q}$ is closed. Then $\mathbb{Q} = \overline{\mathbb{Q}} = \mathbb{R}$ (where the last equality is actually by definition of $\mathbb{R}$, c.f. Theorem 1.36). But this is a contradiction, since f.x. $\sqrt{2} \in \mathbb{R}$, but $\sqrt{2} \notin \mathbb{Q}$.

Assume that $\mathbb{Q}$ is open. Remember that $\{B_{1/k}(x) \mid x \in \mathbb{R}, k \in \mathbb{N}\}$ is a base for $(\mathbb{R}, d_{Eucl})$, where $B_\varepsilon(x) = \{y \in \mathbb{R} \mid d_{Eucl}(x, y) < \varepsilon\}$. Thus $\mathbb{Q} = \bigcup_{j \in J} B_{1/k_j}(x_j)$. This is a contradiction, since every ball $B_\varepsilon(x)$ in $(\mathbb{R}, d_{Eucl})$ also contains irrational numbers.

b) First we have to show, that d is well-defined. Let $x, y, z \in \mathbb{R}$. Then $d(x,y) \geq 0$ and $d(x,y) \leq |x-y| + \sum_{n=1}^{\infty} \frac{1}{2^n} = |x-y| + 1$. Thus d is well-defined.

If $x \neq y$ then $d(x,y) \geq |x-y| > 0$, since $|\cdot|$ is positive definite and $d(x,x) = 0 + \sum_{n=1}^{\infty} \frac{1}{2^n} \min\{1, 0\} = 0$. Thus d is positive definite.

d is symmetric, since $|\cdot|$ is.

And finally d satisfies the triangle inequality:

$$\begin{aligned} &d(x,y) \\ &\quad \leq |x-z| + |z-y| + \sum_{n=1}^{\infty} \frac{1}{2^n} \min\left\{1, \left| \frac{1}{\min_{j\leq n} |x-q_j|} - \frac{1}{\min_{j\leq n} |z-q_j|} \right.\right. \\ &\qquad\qquad \left.\left. + \frac{1}{\min_{j\leq n} |z-q_j|} - \frac{1}{\min_{j\leq n} |y-q_j|} \right| \right\} \\ &\qquad\qquad \leq d(x,z) + d(z,y) \end{aligned}$$

c) Let $q \in \mathbb{Q}$. Then $q = q_k$ for some $k \in \mathbb{N}$. We will show that $\{q_k\} = B^{(d)}_{2^{-k}}(q_k)$, which is open, since $\{B^{(d)}_{1/k}(x) \mid x \in \mathbb{R}, k \in \mathbb{N}\}$, with $B^{(d)}_{\varepsilon}(x) := \{y \in \mathbb{R} \mid d(x,y) < \varepsilon\}$, is (by definition) a base for the by d induced topology of $(\mathbb{R}, d)$. Thus let $y \in \mathbb{R}, y \neq q$.

- Case $y = q_r$ for some $r \in \mathbb{N}$, $r < k$: Then

$$\begin{aligned} d(q,y) = |q_k - q_r| &+ \sum_{n=1}^{r-1} \frac{1}{2^n} \min\left\{1, \left| \frac{1}{\min_{j\leq n} |q_k - q_j|} - \frac{1}{\min_{j\leq n} |q_r - q_j|} \right| \right\} \\ &+ \sum_{n=r}^{k-1} \frac{1}{2^n} \min\{1, \infty\} + \sum_{n=k}^{\infty} \frac{1}{2^n} \min\{1, 0\} \geq \frac{1}{2^k} \end{aligned}$$

- Case $y = q_r$ for some $r \in \mathbb{N}$, $r > k$: Then

$$\begin{aligned} d(q,y) = |q_k - q_r| &+ \sum_{n=1}^{k-1} \frac{1}{2^n} \min\left\{1, \left| \frac{1}{\min_{j\leq n} |q_k - q_j|} - \frac{1}{\min_{j\leq n} |q_r - q_j|} \right| \right\} \\ &+ \sum_{n=k}^{r-1} \frac{1}{2^n} \min\{1, \infty\} + \sum_{n=r}^{\infty} \frac{1}{2^n} \min\{1, 0\} \geq \frac{1}{2^k} \end{aligned}$$

- Case $y \notin \mathbb{Q}$: Then

$$\begin{aligned} d(q,y) = |q_k - y| &+ \sum_{n=1}^{k-1} \frac{1}{2^n} \min\left\{1, \left| \frac{1}{\min_{j\leq n} |q_k - q_j|} - \frac{1}{\min_{j\leq n} |y - q_j|} \right| \right\} \\ &+ \sum_{n=k}^{\infty} \frac{1}{2^n} \min\{1, \infty\} \geq \frac{1}{2^k} \end{aligned}$$

Note, that this lower bound 2^{-k} is independent of y. Thus $d(q,y) \geq 2^{-k}$ and hence $y \notin B^{(d)}_{2^{-k}}(q_k)$ for all $y \in \mathbb{R}$. This proves the claim.

d) We have that $\mathbb{Q} = \bigcup_{j\in\mathbb{N}} \{q_j\}$ is open as the (countable) union of open sets.

$\square$

Problem 4 (Homeomorphisms and completeness).

a) Find two metric spaces that are homeomorphic as topological spaces and such that one is complete whereas the other is not.

   Hint: There exist easy examples. In particular the examples given below will not be accepted as an answer here.

Let d be the metric defined in Problem 3b) restricted to $(\mathbb{R} \setminus \mathbb{Q}) \times (\mathbb{R} \setminus \mathbb{Q})$.

b) Prove that $(\mathbb{R} \setminus \mathbb{Q}, d_{Eucl})$ and $(\mathbb{R} \setminus \mathbb{Q}, d)$ are homeomorphic.

c) Prove that $(\mathbb{R} \setminus \mathbb{Q}, d_{Eucl})$ is not complete whereas $(\mathbb{R} \setminus \mathbb{Q}, d)$ is complete.

[2+4+4 Points]

Proof. a) $(\mathbb{R}, d_{Eucl})$ is known to be a complete metric space. On the other hand the metric space $((-\frac{\pi}{2}, \frac{\pi}{2}), d_{Eucl})$ is not complete, since $(-\frac{\pi}{2}, \frac{\pi}{2})$ is not closed. But $\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ is bijective, continuous and its inverse is continuous, i.e. it is a homeomorphism.

b) to come...

c) to come...

$\square$