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Functional Analysis

Excercise Sheet 3

Remaining solution

Problem 4 (Initial topology)**.** Let $\mathcal{F}$ be a family of functions from a set X to a topological space $(Y, \mathcal{T})$. The $\mathcal{F}$-*initial* topology $\mathcal{T}_i$ on X is the weakest topology such that all functions in $\mathcal{F}$ are continuous.

a) Prove that the family of all finite intersections of sets of the form $f^{-1}(A)$, where $f \in \mathcal{F}$ and $A \in \mathcal{T}_Y$, is a base for $\mathcal{T}_i$.

b) Let $(Z, \mathcal{T}_Z)$ be a topological space and $g : Z \to X$. Prove that g is continuous *iff* for all $f \in \mathcal{F}$ the composition $f \circ g : Z \to Y$ is continuous. (Here, X is equipped with the topology $\mathcal{T}_i$ and Y with $\mathcal{T}$, as above).

c) Let $\{x_n\}_n$ be a sequence in X and $x \in X$. Prove that $x_n \to x$ for $n \to \infty$ in $(X, \mathcal{T}_i)$ *iff* for all $f \in \mathcal{F}$: $f(x_n) \to f(x)$ for $n \to \infty$ in $(Y, \mathcal{T})$.

Application 1: Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and let $P_X : X \times Y \to X$ and $P_Y : X \times Y \to Y$ be the projections on the respective components.

d) Prove that the product topology on $X \times Y$ is the $\mathcal{F}$-initial topology with $\mathcal{F} = \{P_X, P_Y\}$.

   Hint: You have to generalize the above definition of the $\mathcal{F}$-initial topology to different target spaces of the functions in $\mathcal{F}$ in the obvious way.

Application 2: The $\mathcal{F}$-initial topology on $C([0,1])$ with respect to $\mathcal{F} := \{E_x \mid x \in [0,1]\}$, where $E_x : C([0,1]) \to \mathbb{R}, f \mapsto E_x(f) := f(x)$, is called the *topology of pointwise convergence.*

e) Let $\{f_n\}_n$ be a sequence in $C([0,1])$ and $f \in C([0,1])$. Prove that $f_n \to f$ for $n \to \infty$ in the topology of pointwise convergence *iff* for all $x \in [0,1]$: $f_n(x) \to f(x)$ for $n \to \infty$ in the Euclidean topology.

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Proof.

a) Consider $\mathcal{S} := \{f^{-1}(A) \mid f \in \mathcal{F}, A \in \mathcal{T}\}$, $\mathcal{B} := \{\bigcap S \mid S \subseteq \mathcal{S} \text{ finite}\}$, and $\mathcal{T} := \{\bigcup B | B \subseteq \mathcal{B}\}$. Then:

   - $\emptyset \in \mathcal{T}$, with $B = \emptyset \subseteq \mathcal{B}$.
   - $X \in \mathcal{T}$, since $X \in \mathcal{S}$ with $S = \emptyset \subseteq \mathcal{S}$. (*Hint*: $X \in \mathcal{S}$ only if $\mathcal{F} \neq \emptyset$).

- Let $T_1 = \bigcup B_1, T_2 = \bigcup B_2 \in \mathcal{T}$ with $B_1, B_2 \subseteq \mathcal{B}$. Then $B := B_1 \cap B_2 \subseteq \mathcal{B}$ and $T_1 \cap T_2 = \bigcup B_1 \cap \bigcup B_2 = \bigcup B \in \mathcal{B}$.
- $\bigcup T \in \mathcal{T}$ for $T \subseteq \mathcal{T}$ by construction.

Hence $\mathcal{T}$ is a topology on X and $\mathcal{B}$ a base for $\mathcal{T}$ (and $\mathcal{S}$ a subbase). We need to show, that $\mathcal{T} = \mathcal{T}_i$. By construction all $f \in \mathcal{F}$ are continuous in $\mathcal{T}$. Since $\mathcal{T}_i$ is the weakest topology s.t. all $f \in \mathcal{F}$ are continuous we have $\mathcal{T}_i \subseteq T$. Conversely $\mathcal{S} \subseteq \mathcal{T}_i$, since all $f \in \mathcal{F}$ are continuous and $\mathcal{T}_i$ is a topology closed wrt. finite intersections and arbitrary unions. Thus $\mathcal{T} \subseteq \mathcal{T}_i$.

b) "$\Longrightarrow$": Let g be continuous. Let $f \in \mathcal{F}$ and $A \in \mathcal{T}$. Then $f^{-1}(A) \in \mathcal{T}_i$ by definition and $(f \circ g)^{-1}(A) = g^{-1}(f^{-1}(A)) \in \mathcal{T}_Z$, since g continuous. Thus $f \circ g$ continuous.

"$\Longleftarrow$": Let $f \circ g$ be continuous for all $f \in \mathcal{F}$. Let $B \in \mathcal{T}_i$. Since $\mathcal{B}$ is a basis for $\mathcal{T}_i$ (cf. a)), there exists $B_i \in \mathcal{B}$, $i \in I$ s.t. $B = \bigcup_{i \in I} B_i$ and $B_i = \bigcap_{j=1}^{N_i} f_{ij}^{-1}(A_{ij})$, where $f_{ij} \in \mathcal{F}$ and $A_{ij} \in \mathcal{T}$. Then

$$g^{-1}(B) = \bigcup_{i \in I} g^{-1}(B_i) = \bigcup_{i \in I} \bigcap_{j=1}^{N_i} g^{-1}(f_{ij}^{-1}(A_{ij})). \tag{1}$$

Since $g^{-1}(f_{ij}^{-1}(A_{ij})) \in \mathcal{T}_Z$ for all i, j as $g \circ f_{ij}$ is continuous and therefore $g^{-1}(B) \in \mathcal{T}_Z$, since $\mathcal{T}_Z$ is closed wrt. finite intersections and arbitrary unions. Thus g is continuous.

c) "$\Longrightarrow$": Let $x_n \to x$ in $(X, \mathcal{T}_i)$. Let $f \in \mathcal{F}$. Let U be a nbhd of $f(x)$. Then $V := f^{-1}(U) \in \mathcal{T}_i$ by definition of $\mathcal{T}_i$ and $x \in V$. Thus V is a nbhd of x in $(X, \mathcal{T}_i)$. Thus there exists $N \in \mathbb{N}$ s.t. $x_n \in V$ for $n > N$. In particular $f(x_n) \in U$ for $n > N$. Thus $f(x_n) \to f(x)$ in $(Y, \mathcal{T})$.

Alternative: f is continuous and thus sequentially continuous. Thus $f(x_n) \to f(x)$ in $(Y, \mathcal{T})$.

"$\Longleftarrow$": Let $f(x_n) \to f(x)$ in $(Y, \mathcal{T})$ for all $f \in \mathcal{F}$. Let V be a nbhd of x in $(X, \mathcal{T}_i)$. As above $V = \bigcup_{i \in I} B_i$ with $B_i \in \mathcal{B}$ for all $i \in I$ and $x \in B_{i_0}$ for some $i_0 \in I$ with $B_{i_0} = \bigcap_{j=1}^{N} f_j^{-1}(A_j)$, where $f_j \in \mathcal{F}, A_j \in \mathcal{T}$ for all $j = 1..N$. Thus $f_j(x) \in A_j$ and A_j is a nbhd of $f_j(x)$ for all $j = 1..N$. Since $f_j(x_n) \to f_j(x)$ there exists N_j s.t. $f_j(x_n) \in A_j$ for $n > N_j$ and $j = 1..N$. Set $\overline{N} := \max\{N_j \mid j = 1..N\} < \infty$. Then $x_n \in f_j^{-1}(A_j)$ for all $j = 1..N$ and thus $x_n \in \bigcap_{j=1}^{N} f_j^{-1}(A_j) = B_{i_0} \subseteq V$ for $n > \overline{N}$. Thus $x_n \to x$ in $(X, \mathcal{T}_i)$.

d) By 3b) the product topology is the weakest topology s.t. P_X and P_Y are continuous, which is exactly the definition of the $F = \{P_X, P_Y\}$-initial topology.

Hint: Observe, that P_X and P_Y have different target spaces in contrast to the definition of the $\mathcal{F}$-initial topology above. The generalisation is obvious though.

e) By c) we have with $X = C([0, 1])$ and $(Y, \mathcal{T}) = (\mathbb{R}, \mathcal{T}_{Eucl})$ that $f_n \to f$ in $(X, \mathcal{T}_i)$ *iff* for all $x \in [0, 1]$: $f_n(x) = E_x(f_n) \to E_x(f) = f(x)$ in $(Y, \mathcal{T})$.

$\square$

Deadline: May 9, 2016. *For details see http://www.math.lmu.de/~gottwald/16FA/.*