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Functional Analysis

Excercise Sheet 3

Problem 1 (Hausdorff spaces).

a) Let $(X, \mathcal{T})$ be a Hausdorff space and $Y \subseteq X$. Prove that $(Y, \mathcal{T}_Y)$ is a Hausdorff space, where $\mathcal{T}_Y$ is the relative topology of Y wrt. X.

b) Let $X := \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ and $C_r := \{(x,y) \in X \mid r < x^2 + y^2 \leq 1\}$ for $r \in [0,1]$. Prove that $\mathcal{B} := \{C_r \mid r \in [0,1]\} \cup \{X\}$ is a topology on X that is not countable and not Hausdorff.

c) Let $(X, \mathcal{T})$ be a topological space. Prove or disprove: If all *singletons* $\{x\}$ with $x \in X$ are closed, then $(X, \mathcal{T})$ is Hausdorff.

[2+5+3 Points]

Problem 2 (Second countable, separable).

a) Prove that every second countable topological space is separable.

b) Prove that a metric space is second countable *iff* it is separable.

c) Prove that a space equipped with the co-finite topology is separable.

[3+4+3 Points]

Problem 3 (Product topology). Let X_1, X_2, and Z be non-empty topological spaces, and let $X := X_1 \times X_2$ be the product space equipped with the product topology. Let $P_j : X \to X_j, j = 1, 2$ be the projections onto the j-th component, i.e. $P_j x = x_j$ for all $x = (x_1, x_2) \in X$.

a) Prove that P_1 and P_2 are continuous, open, but not necessarily closed maps (i.e. the image of closed sets are *not* necessarily closed).

b) Prove that the product topology on X is the *weakest* topology such that P_1 and P_2 are continuous.

c) Prove that a function $f : Z \to X$ is continuous *iff* each $P_j \circ f : Z \to X_j$, $j = 1, 2$ is continuous.

d) Prove that X is a Hausdorff space *iff* both X_1 and X_2 are Hausdorff.

e) Prove that Z is a Hausdorff space *iff* the "diagonal" $\Delta := \{(z,z) \mid z \in Z\}$ is closed in $Z \times Z$ wrt. the product topology.

[2+2+2+2+2 Points]

Problem 4 (Initial topology). Let $\mathcal{F}$ be a family of functions from a set X to a topological space $(Y, \mathcal{T})$. The $\mathcal{F}$-*initial* topology $\mathcal{T}_i$ on X is the weakest topology such that all functions in $\mathcal{F}$ are continuous.

a) Prove that the family of all finite intersections of sets of the form $f^{-1}(A)$, where $f \in \mathcal{F}$ and $A \in \mathcal{T}_Y$, is a base for $\mathcal{T}_i$.

b) Let $(Z, \mathcal{T}_Z)$ be a topological space and $g : Z \to X$. Prove that g is continuous *iff* for all $f \in \mathcal{F}$ the composition $f \circ g : Z \to Y$ is continuous. (Here, X is equipped with the topology $\mathcal{T}_i$ and Y with $\mathcal{T}$, as above).

c) Let $\{x_n\}_n$ be a sequence in X and $x \in X$. Prove that $x_n \to x$ for $n \to \infty$ in $(X, \mathcal{T}_i)$ *iff* for all $f \in \mathcal{F}$: $f(x_n) \to f(x)$ for $n \to \infty$ in $(Y, \mathcal{T})$.

Application 1: Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and let $P_X : X \times Y \to X$ and $P_Y : X \times Y \to Y$ be the projections on the respective components.

d) Prove that the product topology on $X \times Y$ is the $\mathcal{F}$-initial topology with $\mathcal{F} = \{P_X, P_Y\}$.

   Hint: You have to generalize the above definition of the $\mathcal{F}$-initial topology to different target spaces of the functions in $\mathcal{F}$ in the obvious way.

Application 2: The $\mathcal{F}$-initial topology on $C([0,1])$ with respect to $\mathcal{F} := \{E_x \mid x \in [0,1]\}$, where $E_x : C([0,1]) \to \mathbb{R}, f \mapsto E_x(f) := f(x)$, is called the *topology of pointwise convergence.*

e) Let $\{f_n\}_n$ be a sequence in $C([0,1])$ and $f \in C([0,1])$. Prove that $f_n \to f$ for $n \to \infty$ in the topology of pointwise convergence *iff* for all $x \in [0,1]$: $f_n(x) \to f(x)$ for $n \to \infty$ in the Euclidean topology.

[2+2+2+2+2 Points]

Deadline: May 9, 2016. *For details see http://www.math.lmu.de/~gottwald/16FA/.*